3

Determinants

WEB

INTRODUCTORY EXAMPLE

Determinants in Analytic Geometry

A determinant is a number that is assigned to a square array of numbers in a certain way. This idea was considered as early as 1683 by the Japanese mathematician Seki Takakazu and independently in 1693 by the German mathematician Gottfried Leibniz, about 160 years before a separate theory of matrices developed. For many years, determinants appeared mainly in discussions of systems of linear equations.

In 1750, an article by the Swiss mathematician Gabriel Cramer hinted that determinants might be useful in analytic geometry. In that paper, Cramer used determinants to construct equations of certain curves in the xy-plane. In the same paper, he also produced his famous rule for solving an $n \times n$ system by determinants. Then, in 1812, Augustin-Louis Cauchy published a paper that gave determinantal formulas for volumes of several solid polyhedra, and he connected the formulas with earlier work on determinants. Among the "crystals" Cauchy studied were the tetrahedron in Fig. 1 and the parallelepiped in Fig. 2. If the vertices of the parallelepiped are the origin $\mathbf{0} = (0, 0, 0)$, $\mathbf{v}_1 = (a_1, b_1, c_1)$, $\mathbf{v}_2 = (a_2, b_2, c_2)$, and $\mathbf{v}_3 = (a_3, b_3, c_3)$, then its volume is the absolute value of the determinant of coefficient matrix of the system:

$$
\begin{aligned}
a_1x + b_1y + c_1z &= 0 \\
a_2x + b_2y + c_2z &= 0 \\
a_3x + b_3y + c_3z &= 0
\end{aligned}
$$

Cauchy's use of determinants in analytic geometry stimulated an intense interest in applications of determinants that lasted for about 100 years. A mere summary of what was known by the early 1900s filled a four-volume treatise by Thomas Muir.

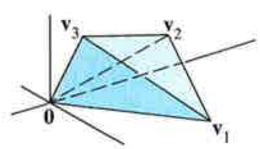

FIGURE 1 A tetrahedron.

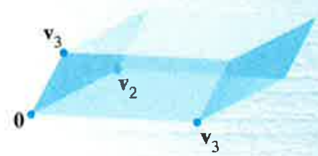

FIGURE 2 A parallelepiped.

In Cauchy's day, when life was simple and matrices were small, determinants played a major role in analytic geometry and other parts of mathematics. Today, determinants are of little numerical value in the large-scale matrix computations that occur so often. Nevertheless, determinantal formulas still give important information about matrices, and a knowledge of determinants is useful in some applications of linear algebra.

We have three goals in this chapter: to prove an invertibility criterion for a square matrix A that involves the entries of A rather than its columns, to give formulas for A^{-1} and $A^{-1}\mathbf{b}$ that are used in theoretical applications, and to derive the geometric interpretation of a determinant described in the chapter introduction. The first goal is reached in Section 3.2 and the other two in Section 3.3.

3.1 INTRODUCTION TO DETERMINANTS

Recall from Section 2.2 that a 2×2 matrix is invertible if and only if its determinant is nonzero. To extend this useful fact to larger matrices, we need a definition for the determinant of an $n\times n$ matrix. We can discover the definition for the 3×3 case by watching what happens when an invertible 3×3 matrix A is row reduced.

Consider $A=[a_{ij}]$ with $a_{11}\neq 0$. If we multiply the second and third rows of A by a_{11} and then subtract appropriate multiples of the first row from the other two rows, we find that A is row equivalent to the following two matrices:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & a_{11}a_{32}-a_{12}a_{31} & a_{11}a_{33}-a_{13}a_{31} \end{bmatrix} \tag{1}$$

Since A is invertible, either the $(2,2)$-entry or the $(3,2)$-entry on the right in (1) is nonzero. Let us suppose that the $(2,2)$-entry is nonzero. (Otherwise, we can make a row interchange before proceeding.) Multiply row 3 by $a_{11}a_{22}-a_{12}a_{21}$, and then to the new row 3 add $-(a_{11}a_{32}-a_{12}a_{31})$ times row 2. This will show that

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

where

$$\Delta = a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31} \tag{2}$$

Since A is invertible, Δ must be nonzero. The converse is true, too, as we will see in Section 3.2. We call Δ in (2) the **determinant** of the 3×3 matrix A.

Recall that the determinant of a 2×2 matrix, $A=[a_{ij}]$, is the number

$$\det A = a_{11}a_{22}-a_{12}a_{21}$$

For a 1×1 matrix—say, $A=[a_{11}]$—we define $\det A = a_{11}$. To generalize the definition of the determinant to larger matrices, we'll use 2×2 determinants to rewrite the 3×3 determinant Δ described above. Since the terms in Δ can be grouped as

$$(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}),$$

$$\Delta = a_{11}\cdot\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\cdot\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\cdot\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

For brevity, we write

$$\Delta = a_{11}\cdot\det A_{11} - a_{12}\cdot\det A_{12} + a_{13}\cdot\det A_{13} \tag{3}$$

where A_{11}, A_{12}, and A_{13} are obtained from A by deleting the first row and one of the three columns. For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A. For instance, if

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

then A_{32} is obtained by crossing out row 3 and column 2,

$$\begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

so that

$$A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

We can now give a *recursive* definition of a determinant. When $n = 3$, $\det A$ is defined using determinants of the 2×2 submatrices A_{1j}, as in (3) above. When $n = 4$, $\det A$ uses determinants of the 3×3 submatrices A_{1j}. In general, an $n\times n$ determinant is defined by determinants of $(n-1)\times(n-1)$ submatrices.

DEFINITION

For $n \geq 2$, the **determinant** of an $n\times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}\det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}a_{1n}\det A_{1n} \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j} \end{aligned}$$

EXAMPLE 1 Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution Compute $\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$:

$$\det A = 1 \cdot \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$
$$= 1(0-2) - 5(0-0) + 0(-4-0) = -2$$

Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets. Thus the calculation in Example 1 can be written as

$$\det A = 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

To state the next theorem, it is convenient to write the definition of $\det A$ in a slightly different form. Given $A = [a_{ij}]$, the **(i, j)-cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij} \tag{4}$$

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

This formula is called a **cofactor expansion across the first row** of A. We omit the proof of the following fundamental theorem to avoid a lengthy digression.

THEOREM 1

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in (4) is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

The plus or minus sign in the (i, j)-cofactor depends on the position of a_{ij} in the matrix, regardless of the sign of a_{ij} itself. The factor $(-1)^{i+j}$ determines the following checkerboard pattern of signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

EXAMPLE 2 Use a cofactor expansion across the third row to compute $\det A$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Solution Compute

$$\begin{aligned}\det A &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}\\ &= (-1)^{3+1}a_{31}\det A_{31} + (-1)^{3+2}a_{32}\det A_{32} + (-1)^{3+3}a_{33}\det A_{33}\\ &= 0\begin{vmatrix}5 & 0\\ 4 & -1\end{vmatrix} - (-2)\begin{vmatrix}1 & 0\\ 2 & -1\end{vmatrix} + 0\begin{vmatrix}1 & 5\\ 2 & 4\end{vmatrix}\\ &= 0 + 2(-1) + 0 = -2\end{aligned}$$

Theorem 1 is helpful for computing the determinant of a matrix that contains many zeros. For example, if a row is mostly zeros, then the cofactor expansion across that row has many terms that are zero, and the cofactors in those terms need not be calculated. The same approach works with a column that contains many zeros.

EXAMPLE 3 Compute $\det A$, where

$$A = \begin{bmatrix}3 & -7 & 8 & 9 & -6\\ 0 & 2 & -5 & 7 & 3\\ 0 & 0 & 1 & 5 & 0\\ 0 & 0 & 2 & 4 & -1\\ 0 & 0 & 0 & -2 & 0\end{bmatrix}$$

Solution The cofactor expansion down the first column of A has all terms equal to zero except the first. Thus

$$\det A = 3\cdot\begin{vmatrix}2 & -5 & 7 & 3\\ 0 & 1 & 5 & 0\\ 0 & 2 & 4 & -1\\ 0 & 0 & -2 & 0\end{vmatrix} - 0\cdot C_{21} + 0\cdot C_{31} - 0\cdot C_{41} + 0\cdot C_{51}$$

Henceforth we will omit the zero terms in the cofactor expansion. Next, expand this 4×4 determinant down the first column, in order to take advantage of the zeros there. We have

$$\det A = 3\cdot 2\cdot\begin{vmatrix}1 & 5 & 0\\ 2 & 4 & -1\\ 0 & -2 & 0\end{vmatrix}$$

This 3×3 determinant was computed in Example 1 and found to equal -2. Hence $\det A = 3\cdot 2\cdot(-2) = -12$.

The matrix in Example 3 was nearly triangular. The method in that example is easily adapted to prove the following theorem.

THEOREM 2 If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

The strategy in Example 3 of looking for zeros works extremely well when an entire row or column consists of zeros. In such a case, the cofactor expansion along such a row or column is a sum of zeros! So the determinant is zero. Unfortunately, most cofactor expansions are not so quickly evaluated.

NUMERICAL NOTE

By today's standards, a 25×25 matrix is small. Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion. In general, a cofactor expansion requires over $n!$ multiplications, and 25! is approximately 1.5×10^{25}.

If a computer performs one trillion multiplications per second, it would have to run for over 500,000 years to compute a 25×25 determinant by this method. Fortunately, there are faster methods, as we'll soon discover.

Exercises 19–38 explore important properties of determinants, mostly for the 2×2 case. The results from Exercises 33–36 will be used in the next section to derive the analogous properties for $n \times n$ matrices.

PRACTICE PROBLEM

Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$.

3.1 EXERCISES

Compute the determinants in Exercises 1–8 using a cofactor expansion across the first row. In Exercises 1–4, also compute the determinant by a cofactor expansion down the second column.

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$

5. $\begin{vmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{vmatrix}$

6. $\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$

7. $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$

Compute the determinants in Exercises 9–14 by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

9. $\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$

10. $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$

11. $\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

12. $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{vmatrix}$

13. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$

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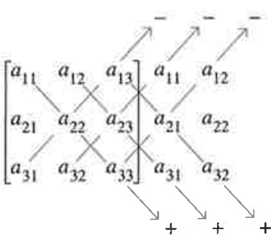
14. $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

The expansion of a 3×3 determinant can be remembered by the following device. Write a second copy of the first two columns to the right of the matrix, and compute the determinant by multiplying entries on six diagonals:



Add the downward diagonal products and subtract the upward products. Use this method to compute the determinants in Exercises 15–18. ***Warning:*** *This trick does not generalize in any reasonable way to* 4×4 *or larger matrices.*

15. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$

16. $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$

17. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$

18. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$

In Exercises 19–24, explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

20. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

21. $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix}$

22. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

23. $\begin{bmatrix} 1 & 1 & 1 \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}, \begin{bmatrix} k & k & k \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}$

24. $\begin{bmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{bmatrix}$

Compute the determinants of the elementary matrices given in Exercises 25–30. (See Section 2.2.)

25. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$

27. $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$

29. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

30. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Use Exercises 25–28 to answer the questions in Exercises 31 and 32. Give reasons for your answers.

31. What is the determinant of an elementary row replacement matrix?

32. What is the determinant of an elementary scaling matrix with k on the diagonal?

In Exercises 33–36, verify that $\det EA = (\det E)(\det A)$, where E is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

35. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

37. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

38. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

In Exercises 39 and 40, A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

39. a. An $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.
    b. The (i, j)-cofactor of a matrix A is the matrix A_{ij} obtained by deleting from A its ith row and jth column.

40. a. The cofactor expansion of $\det A$ down a column is the negative of the cofactor expansion along a row.
    b. The determinant of a triangular matrix is the sum of the entries on the main diagonal.

41. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{0}$, and compute the determinant of $[\mathbf{u} \;\; \mathbf{v}]$. How do they compare? Replace the

first entry of $\mathbf{v}$ by an arbitrary number x, and repeat the problem. Draw a picture and explain what you find.

42. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$, where a, b, c are positive (for simplicity). Compute the area of the parallelogram determined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinants of the matrices $[\mathbf{u} \quad \mathbf{v}]$ and $[\mathbf{v} \quad \mathbf{u}]$. Draw a picture and explain what you find.

43. **[M]** Is it true that $\det(A + B) = \det A + \det B$? To find out, generate random 5×5 matrices A and B, and compute $\det(A + B) - \det A - \det B$. (Refer to Exercise 37 in Section 2.1.) Repeat the calculations for three other pairs of $n\times n$ matrices, for various values of n. Report your results.

44. **[M]** Is it true that $\det AB = (\det A)(\det B)$? Experiment with four pairs of random matrices as in Exercise 43, and make a conjecture.

45. **[M]** Construct a random 4×4 matrix A with integer entries between -9 and 9, and compare $\det A$ with $\det A^T$, $\det(-A)$, $\det(2A)$, and $\det(10A)$. Repeat with two other random 4×4 integer matrices, and make conjectures about how these determinants are related. (Refer to Exercise 36 in Section 2.1.) Then check your conjectures with several random 5×5 and 6×6 integer matrices. Modify your conjectures, if necessary, and report your results.

46. **[M]** How is $\det A^{-1}$ related to $\det A$? Experiment with random $n\times n$ integer matrices for $n = 4$, 5, and 6, and make a conjecture. *Note:* In the unlikely event that you encounter a matrix with a zero determinant, reduce it to echelon form and discuss what you find.

SOLUTION TO PRACTICE PROBLEM

Take advantage of the zeros. Begin with a cofactor expansion down the third column to obtain a 3×3 matrix, which may be evaluated by an expansion down its first column.

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix} = (-1)^{1+3}2\begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

$$= 2\cdot(-1)^{2+1}(-5)\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 20$$

The $(-1)^{2+1}$ in the next-to-last calculation came from the (2, 1)-position of the -5 in the 3×3 determinant.

3.2 PROPERTIES OF DETERMINANTS

The secret of determinants lies in how they change when row operations are performed. The following theorem generalizes the results of Exercises 19–24 in Section 3.1. The proof is at the end of this section.

THEOREM 3 **Row Operations**

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k\cdot\det A$.

The following examples show how to use Theorem 3 to find determinants efficiently.

EXAMPLE 1 Compute det A, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.

Solution The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$$

An interchange of rows 2 and 3 reverses the sign of the determinant, so

$$\det A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

A common use of Theorem 3(c) in hand calculations is to *factor out a common multiple of one row* of a matrix. For instance,

$$\begin{vmatrix} * & * & * \\ 5k & -2k & 3k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ 5 & -2 & 3 \\ * & * & * \end{vmatrix}$$

where the starred entries are unchanged. We use this step in the next example.

EXAMPLE 2 Compute det A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Solution To simplify the arithmetic, we want a 1 in the upper-left corner. We could interchange rows 1 and 4. Instead, we factor out 2 from the top row, and then proceed with row replacements in the first column:

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

Next, we could factor out another 2 from row 3 or use the 3 in the second column as a pivot. We choose the latter operation, adding 4 times row 2 to row 3:

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

Finally, adding $-1/2$ times row 3 to row 4, and computing the "triangular" determinant, we find that

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2\cdot(1)(3)(-6)(1) = -36$$

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges. (This is always possible. See the row reduction algorithm of Section 1.2.) If there are r interchanges, then Theorem 3 shows that

$$\det A = (-1)^r \det U$$

Since U is in echelon form, it is triangular, and so $\det U$ is the product of the diagonal entries $u_{11}, \ldots, u_{nn}$. If A is invertible, the entries u_{ii} are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1's). Otherwise, at least u_{nn} is zero, and the product $u_{11} \cdots u_{nn}$ is zero. See Fig. 1. Thus

$$\det A = \begin{cases} (-1)^r \cdot \begin{pmatrix}\text{product of} \\ \text{pivots in } U\end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases} \quad (1)$$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\det U \neq 0$

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det U = 0$

FIGURE 1
Typical echelon forms of square matrices.

It is interesting to note that although the echelon form U described above is not unique (because it is not completely row reduced), and the pivots are not unique, the *product* of the pivots *is* unique, except for a possible minus sign.

Formula (1) not only gives a concrete interpretation of $\det A$ but also proves the main theorem of this section:

THEOREM 4 A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 4 adds the statement "$\det A \neq 0$" to the Invertible Matrix Theorem. A useful corollary is that $\det A = 0$ when the columns of A are linearly dependent. Also, $\det A = 0$ when the *rows* of A are linearly dependent. (Rows of A are columns of A^T, and linearly dependent columns of A^T make A^T singular. When A^T is singular, so is A, by the Invertible Matrix Theorem.) In practice, linear dependence is obvious only when two columns or two rows are the same or a column or a row is zero.

EXAMPLE 3 Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

Solution Add 2 times row 1 to row 3 to obtain

$$\det A = \det \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

because the second and third rows of the second matrix are equal.

NUMERICAL NOTES

1. Most computer programs that compute $\det A$ for a general matrix A use the method of formula (1) above.
2. It can be shown that evaluation of an $n \times n$ determinant using row operations requires about $2n^3/3$ arithmetic operations. Any modern microcomputer can calculate a 25×25 determinant in a fraction of a second, since only about 10,000 operations are required.

CD Determinants and Flops

Computers can also handle large "sparse" matrices, with special routines that take advantage of the presence of many zeros. Of course, zero entries can speed hand computations, too. The calculations in the next example combine the power of row operations with the strategy from Section 3.1 of using zero entries in cofactor expansions.

EXAMPLE 4 Compute $\det A$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$.

Solution A good way to begin is to use the 2 in column 1 as a pivot, eliminating the -2 below it. Then use a cofactor expansion to reduce the size of the determinant, followed by another row replacement operation. Thus

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

An interchange of rows 2 and 3 would produce a "triangular determinant." Another approach is to make a cofactor expansion down the first column:

$$\det A = (-2)(1) \begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix} = -2 \cdot (15) = -30$$

Column Operations

We can perform operations on the columns of a matrix in a way that is analogous to the row operations we have considered. The next theorem shows that column operations have the same effects on determinants as row operations.

THEOREM 5 If A is an $n \times n$ matrix, then $\det A^T = \det A$.

PROOF The theorem is obvious for $n = 1$. Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$. Then the cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T, because the cofactors involve $k \times k$ determinants. Hence the cofactor expansion of $\det A$ along the first *row* equals the cofactor expansion of $\det A^T$ down the first *column*. That is, A and A^T have equal determinants. Thus the theorem is true for $n = 1$, and the truth of the theorem for one value of n implies its truth for the next value of n. By the principle of induction, the theorem is true for all $n \geq 1$. ■

Because of Theorem 5, each statement in Theorem 3 is true when the word *row* is replaced everywhere by *column*. To verify this property, one merely applies the original Theorem 3 to A^T. A row operation on A^T amounts to a column operation on A.

Column operations are useful for both theoretical purposes and hand computations. However, for simplicity we'll perform only row operations in numerical calculations.

Determinants and Matrix Products

The proof of the following useful theorem is at the end of the section. Applications are in the exercises.

THEOREM 6 Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

EXAMPLE 5 Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

Solution

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

and

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

Since $\det A = 9$ and $\det B = 5$,

$$(\det A)(\det B) = 9 \cdot 5 = 45 = \det AB$$

■

Warning: A common misconception is that Theorem 6 has an analogue for *sums* of matrices. However, $\det(A+B)$ is *not* equal to $\det A + \det B$, in general.

A Linearity Property of the Determinant Function

For an $n \times n$ matrix A, we can consider $\det A$ as a function of the n column vectors in A. We will show that if all columns except one are held fixed, then $\det A$ is a *linear function* of that one (vector) variable.

Suppose that the jth column of A is allowed to vary, and write

$$A = [\,\mathbf{a}_1 \;\; \cdots \;\; \mathbf{a}_{j-1} \;\; \mathbf{x} \;\; \mathbf{a}_{j+1} \;\; \cdots \;\; \mathbf{a}_n\,]$$

Define a transformation T from $\mathbb{R}^n$ to $\mathbb{R}$ by

$$T(\mathbf{x}) = \det\,[\,\mathbf{a}_1 \;\; \cdots \;\; \mathbf{a}_{j-1} \;\; \mathbf{x} \;\; \mathbf{a}_{j+1} \;\; \cdots \;\; \mathbf{a}_n\,]$$

Then,

$$T(c\mathbf{x}) = cT(\mathbf{x}) \quad \text{for all scalars } c \text{ and all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2}$$

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \text{ in } \mathbb{R}^n \tag{3}$$

Property (2) is Theorem 3(c) applied to the columns of A. A proof of property (3) follows from a cofactor expansion of $\det A$ down the jth column. (See Exercise 43.) This (multi-) linearity property of the determinant turns out to have many useful consequences that are studied in more advanced courses.

Proofs of Theorems 3 and 6

It is convenient to prove Theorem 3 when it is stated in terms of the elementary matrices discussed in Section 2.2. We call an elementary matrix E a *row replacement* (*matrix*) if E is obtained from the identity I by adding a multiple of one row to another row; E is an *interchange* if E is obtained by interchanging two rows of I; and E is *a scale by* r if E is obtained by multiplying a row of I by a nonzero scalar r. With this terminology, Theorem 3 can be reformulated as follows:

If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det EA = (\det E)(\det A)$$

where

$$\det E = \begin{cases} 1 & \text{if } E \text{ is a row replacement} \\ -1 & \text{if } E \text{ is an interchange} \\ r & \text{if } E \text{ is a scale by } r \end{cases}$$

PROOF OF THEOREM 3 The proof is by induction on the size of A. The case of a 2×2 matrix was verified in Exercises 33–36 of Section 3.1. Suppose the theorem has been verified for determinants of $k \times k$ matrices with $k \geq 2$, let $n = k+1$, and let A be $n \times n$.

The action of E on A involves either two rows or only one row. So we can expand $\det EA$ across a row that is unchanged by the action of E, say, row i. Let A_{ij} (respectively, B_{ij}) be the matrix obtained by deleting row i and column j from A (respectively, EA). Then the rows of B_{ij} are obtained from the rows of A_{ij} by the same type of elementary row operation that E performs on A. Since these submatrices are only $k \times k$, the induction assumption implies that

$$\det B_{ij} = \alpha \cdot \det A_{ij}$$

where $\alpha = 1, -1$, or r, depending on the nature of E. The cofactor expansion across row i is

$$\begin{aligned}\det EA &= a_{i1}(-1)^{i+1} \det B_{i1} + \cdots + a_{in}(-1)^{i+n} \det B_{in} \\ &= \alpha a_{i1}(-1)^{i+1} \det A_{i1} + \cdots + \alpha a_{in}(-1)^{i+n} \det A_{in} \\ &= \alpha \cdot \det A\end{aligned}$$

In particular, taking $A = I_n$, we see that $\det E = 1, -1$, or r, depending on the nature of E. Thus the theorem is true for $n = 2$, and the truth of the theorem for one value of n implies its truth for the next value of n. By the principle of induction, the theorem must be true for $n \geq 2$. The theorem is trivially true for $n = 1$. ■

PROOF OF THEOREM 6 If A is not invertible, then neither is AB, by Exercise 27 in Section 2.3. In this case, $\det AB = (\det A)(\det B)$, because both sides are zero, by Theorem 4. If A is invertible, then A and the identity matrix I_n are row equivalent by the Invertible Matrix Theorem. So there exist elementary matrices $E_1, \ldots, E_p$ such that

$$A = E_p E_{p-1} \cdots E_1 \cdot I_n = E_p E_{p-1} \cdots E_1$$

For brevity, write $|A|$ for $\det A$. Then repeated application of Theorem 3, as rephrased above, shows that

$$\begin{aligned}|AB| &= |E_p \cdots E_1 B| = |E_p||E_{p-1} \cdots E_1 B| = \cdots \\ &= |E_p| \cdots |E_1||B| = \cdots = |E_p \cdots E_1||B| \\ &= |A||B|\end{aligned}$$ ■

PRACTICE PROBLEMS

1. Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ in as few steps as possible.

2. Use a determinant to decide if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, when

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}.$$