Review Sheet for Final Exam

M27 – Linear Algebra   Spring 2013   Swarthmore College   Prof. Scott Cook
May 17, 2013   7:00-10:00PM   Science Center 199

- This is a work in progress. It may be amended at any moment, up until the start of the exam. See timestamp in footer. This list is NOT COMPLETE. Any topic covered in lecture or homework is exam eligible unless otherwise explicitly ruled out here. If a topic has been not been discussed here and you would like to inquire of its exam eligibility, please post to the forum for this exam on moodle.
- A high-efficiency study method is to rework old homework problems, trying to forget what you did before. Though the exam is unlikely to have a problem from homework verbatim, if I thought a certain style of problem was important enough to assign once, there is a better than random chance that I will think it is still important at exam time.
- I have posted exams 1 and 2 from this semester and from last semester. I expect to write an exam that is similar *in format*. Do *not* expect problem that are similar *in content*.
- As stated in the syllabus, "Exams will be mostly free response and will contain some purely conceptual problems, some mostly computational problems, and some problems which require substantial conceptual **and** computational facility."
- As stated in the syllabus, "You may use scientific (not graphing) calculators on exams. Nothing that can handle matrices."

Chapter 1

Topics to know

- Word problems - Turn the description of a system into mathematical expression(s)
- Solve a system of n equations involving m variables using matrices and row operations
- Determine all solutions of a system. How many are there? (0,1,a lines worth/1 free variable, a planes worth/2 free variables, etc).
- What is rref(A)? (state definition)
- Given A, find rref(A) and use it to determine whether $A\vec{x} = \vec{b}$ must have a unique solution for any $\vec{b}$. If not, determine how the number of solutions to $A\vec{x} = \vec{b}$ could vary between different $\vec{b}$ (for example, perhaps some $\vec{b}$ have no solution while other $\vec{b}$ have a plane's worth of solutions, etc).
- Compute rank(A).
- Basic matrix algebra operations and the geometric meaning
- Definition of a linear combination
- Describe 3 views of $A\vec{x}$

Proofs to know

- Prove basic facts about matrix algebra (such as the list of 5 I gave in lecture and you proved some of in homework).

Not eligible for the exam

- Nothing I have thought of so far. Feel free to ask.

Chapter 2

Section 2.1

Topics to know

- What does it mean for a transformation to be injective, surjective, or bijective? (state definitions)
- Determine whether a transformation has these properties.

- What does it mean for a transformation to be invertible?
- What does it mean for a transformation to be linear? (state definition - 2 properties). (recall that Bretscher approached this differently than our lecture) Be able to show whether a given transformation is linear using these 2 properties.
- Know that T is linear if and only if there is a matrix that does it. (no proof)
- If T is linear, what are the columns of the matrix that does it?

Proofs to know

- Prove that T is linear if and only if there is a matrix that does it.

Not eligible for the exam

- Nothing I have thought of so far. Feel free to ask.

Section 2.2

Topics to know

- Know the 5 types of 2 dimensional transformations we discussed. For each, know
  i) What it does geometrically
  ii) For all 5, that it is linear using a geometric argument (a well drawn and explained picture)
  iii) For scaling, orthogonal projection, and reflection, show that it is linear using an algebraic argument
  iv) What matrix does it?
  v) **I will give you the formula for orthogonal projection in Theorem 5.1.5**
  vi) How to use this knowledge to solve problems

Proofs to know

- See above

Not eligible for the exam

- Don't memorize the formula for orthogonal projection

Section 2.3

Topics to know

- How do we multiply two matrices? (state definition and perform the computation)
- Why do we define matrix multiplication this way? (Answer - So that composing two linear transformations is equivalent to multiplying their matrices. Get the order right; the order you write the symbols for matrix multiplication is the opposite of the order you draw the blob diagram.)
- Given 2 matrices, determine if they can be multiplied. If so, do it.

Proofs to know

- Prove that the composition of 2 linear transformations is linear (verify the 2 properties that define linearity without referring to any matrices)

Not eligible for the exam

- Proofs of theorems 2.3.5-2.3.8
- Block matrices

Section 2.4

Topics to know

- Know that a transformation T is invertible if and only if it is bijective.
- Know several ways to check if an $n \times n$ matrix is invertible.
- If we know a linear transformation is invertible, how do we find its inverse? (Answer - Find its matrix A, augment with I_n, row reduce.)

Proofs to know

- Show that $(BA)^{-1} = A^{-1}B^{-1}$ using a blob diagram

Not eligible for the exam

- p.85-88

Chapter 3

Section 3.1

Topics to know

- What are ker(T) and im(T)? (state definition)
- Why do we care about ker(T) and im(T)? (In other words, how do ker(T) and im(T) relate to injectivity and surjectivity?)

Proofs to know

- Prove that T is injective **if and only if** ker(T) contains only the zero vector
- Prove that ker(T) and im(T) are subspaces

Not eligible for the exam

- Error correcting codes (3.1#53,54)

Section 3.2

Topics to know

- What is a subspace? (state definition - 3 properties)
- What is the geometric meaning of the 3 properties that define a subspace?
- What are the types of subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$? How about $\mathbb{R}^n$?
- What does it mean for a set of vectors to be linearly independent? (state definition)
- Given a set of vectors, determine whether it is a linearly independent set. (We discussed several different methods to approach this problem). Clearly explain why or why not.
- What is the span of a set of vectors? (definition)
- Given a set of vectors, determine its span. Determine whether another vector or subspace lies in the span. Clearly explain why or why not.
- What is a basis? (state definition)
- Together, the properties that define a basis say that the set is as small as possible, but not smaller. Explain this statement.
- Why do we care about bases? (Answer - Most efficient way to describe the subspace; for any vector in the subspace, there is one and only one way to describe it using the basis. This is the idea of the coordinates of a vector with respect to the basis)

Proofs to know

- State and prove Thm 3.2.10

Not eligible for the exam

- Nothing I have thought of so far. Feel free to ask.

Section 3.3

Topics to know

- What is the dimension of a subspace? (state definition - it must contain the word "any".)
- Why does the definition of dimension make sense? (Why can the definition above use the word "any"?)
- Find a basis for ker(T) and im(T).
- Use rank-nullity theorem to solve problems about dimension

Proofs to know

- The following were all done carefully in lecture
- State and prove Thm 3.3.2
- State and prove Thm 3.3.6
- State and prove Thm 3.3.7

Not eligible for the exam

- Nothing I have thought of so far. Feel free to ask.

Section 3.4

Topics to know

- Know that there are many bases for a subspace.
- Given a vector $\vec{x}$, there is one way to express it using each basis $\mathfrak{B}$. Find the $\mathfrak{B}$-coordinates of $\vec{x}$.
- Given a linear transformation T, there is one way to express it using each basis $\mathfrak{B}$. Find the $\mathfrak{B}$-matrix of T.
- Know what is means for two matrices to be similar. (Definition 3.4.5 AND the sentence directly below it). Also know Thm 7.3.6 for some of the properties that similar matrices have in common.

Proofs to know

- Nothing I have thought of so far. Feel free to ask.

Not eligible for the exam

- Nothing I have thought of so far. Feel free to ask.

Chapter 5

Section 5.1

Topics to know

- What does it mean for a vector to be orthogonal to another vector? (state definition)
- What does it mean for a vector to be orthogonal to a subspace? (state definition)
- What does it mean for a set of vectors to be orthonormal? (state definition)

- Determine whether given vectors/subspaces are orthogonal/orthonormal. Clearly explain why or why not.
- Compute $\text{proj}_V(\vec{\boldsymbol{x}})$, the orthogonal projection of a vector $\vec{\boldsymbol{x}}$ onto a subspace V. (I will give you the formula.)
- Why do we care about $\vec{\boldsymbol{x}}^{||} := \text{proj}_V(\vec{\boldsymbol{x}})$? (Answer - it is the vector in V most like $\vec{\boldsymbol{x}}$).
- Why do we care about $\vec{\boldsymbol{x}}^{\perp} := \vec{\boldsymbol{x}} - \text{proj}_V(\vec{\boldsymbol{x}})$? (Answer - its length is how much we changed $\vec{\boldsymbol{x}}$ to get into V; the error we induce by projection).
- What is the orthogonal complement $V^{\perp}$ of a subspace V?

Proofs to know

- Nothing I have thought of so far. Feel free to ask.

Not eligible for the exam

- I will supply the formula for orthogonal projection in Thm 5.1.5. Do not memorize it. You need to know how to use it.
- **Proofs from chapter 5**
- p.194-199

Section 5.2

Topics to know

- What is the Gram-Schmidt process? (describe the algorithm)
- Why do we care about the Gram-Schmidt process? (It turns any basis into an orthonormal basis.)
- Do Gram-Schmidt for 3 vectors. (I will try to keep the computational complexity low)
- Regarding QR-factorization, you do not need to know very much; only know that it helps us find expansion factors - see section 6.3 below.

Proofs to know

- Nothing I have thought of so far. Feel free to ask.

Not eligible for the exam

- **Proofs from chapter 5**

Section 5.3

skip

Section 5.4

Topics to know

- Suppose you have a set of data points and a type of function to fit to that data. Describe how you do this using the tools of chapter 5. I will probably not ask you to completely carry out such a calculation (as it would take too long), but I may ask you to do parts and describe the remainder of the process.
- Explain why it is usually impossible to solve $A\vec{\boldsymbol{x}}^* = \vec{\boldsymbol{b}}$ when A and $\vec{\boldsymbol{b}}$ come from measured data. What do you do when this happens? You solve $A\vec{\boldsymbol{x}}^* = \text{proj}_{\text{im}(A)}\vec{\boldsymbol{b}}$. Why is this the right way to change the problem?
- Know that, with some substantial extra thinking, the formula $A\vec{\boldsymbol{x}}^* = \text{proj}_{\text{im}(A)}\vec{\boldsymbol{b}}$ can be refined to the normal equation $A^TA\vec{\boldsymbol{x}}^* = A^T\vec{\boldsymbol{b}}$. You do not need to know how to derive the normal equation, but you do need to know how to use it.

Proofs to know

- Nothing I have thought of so far. Feel free to ask.

Not eligible for the exam

- ~~**Most of this section**~~ **. . . I don't know how this item got here, but this section is definitely eligible.**
- **Proofs from chapter 5**

Section 5.5

skip

Chapter 6

Section 6.1-6.2

Topics to know

- Know that A is invertible if and only if $\det A \neq 0$ (This is theorem 6.2.4)
- Compute the determinant of a matrix using the cofactor expansion definition.
- Compute the determinant of a matrix using helpful facts such as Theorems 6.1.4v1, 6.1.4v2, 6.1.5, 6.2.1, 6.2.3, 6.2.6, 6.2.7, 6.2.8 (no proofs)

Proofs to know

- State and prove Thm 6.2.4 (you may use the other theorems from 6.1, 6.2, and earlier material without proving them)
- Prove Theorem 6.1.4v1 using induction

Not eligible for the exam

- The permutations/patterns and axiomatic definitions of the determinant
- Proofs from these sections not mentioned above

Section 6.3

Topics to know

- Find area/volumes of shapes using the determinant
- Find the expansion factor of a linear transformation T using the determinant

Proofs to know

- To see that $|\det A|$ equals the expansion factor of A, we needed 3 steps. You need to know what these 3 steps are and how they relate to each. See below.
- First, we noted that the expansion factor for a LINEAR transformation (defined on p.281) is the same for all regions in the domain. So, we choose to consider the region defined by the standard basis vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$. It has volume 1. It's image is the m-parallelepiped defined by the columns of A. Thus the expansion factor equals its volume.

  Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be the column vectors of A. Consider the expression $||\vec{v}_1|| \cdot ||\vec{v}_2^{\perp}|| \cdots ||\vec{v}_m^{\perp}||$.
- Second, we saw that this expression equals the volume of the m-parallelepiped defined by the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$. Be able to explain clearly why this is true. In other words, clearly explain how the formula found in the middle of p.280 relates to figures 1 and 2 on p.278 and 279. This step is fully explained in section 6.3.
- Third, we saw that this expression also equals $|\det A|$. This step is primarily explained in section 5.2; it comes from the QR-factorization of A. You do not need to be able to provide a detailed proof of this step; just know what the step is and that it comes from the QR-factorization.

Not eligible for the exam

- Cramer's rule

Chapter 7

Section 7.1

Topics to know

- How to encode the rules that govern a system into a matrix (such as the lilac bush example and 7.1#52-54).
- What are eigenvectors and eigenvalues? (definition)
- What does it mean geometrically for a vector to be an eigenvector?
- Why do we care about eigenvectors and eigenvalues? (They help us know our future)
- Know that the zero vector $\vec{\mathbf{0}}$ can NOT be an eigenvector but the number 0 can be an eigenvalue. Can you think of a transformation in section 2.2 that has an eigenvector with eigenvalue 0?
- Know how to predict the future of a linear system using the algorithm discussed repeatedly in lecture.
- A Markov chain is a special type of dynamical system (the matrix is stochastic). It is useful when the "total amount of stuff" does not change; it just gets redistributed. Use Markov chains to analyze systems (for example, the social network and Yatzi homeworks).
- Summary 7.1.5 on p.305. Understand each of the 11 characterizations of invertibility.

Proofs to know

- Nothing I have thought of so far. Feel free to ask.

Not eligible for the exam

- Nothing I have thought of so far. Feel free to ask.

Section 7.2

Topics to know

- Find eigenvalues (no complex eigenvalues/eigenvectors)
- What is the algebraic multiplicity of an eigenvalue? (state definition)
- What is the characteristic polynomial of a matrix? (state definition)
- Why do we care about the characteristic polynomial? (Answer - its root are the eigenvalues of the matrix)
- What is the trace of a matrix? (state definition)
- Know that det(A)=product of eigenvalues and tr(A)=sum of eigenvalues (no proof)

Proofs to know

- Nothing I have thought of so far. Feel free to ask.

Not eligible for the exam

- Complex eigenvalues/eigenvectors

Section 7.3

Topics to know

- **What is an eigenspace? (state definition 7.3.1. One note - in lecture we said that E_λ is essentially the set of the eigenvectors with eigenvalue λ. We also said in section 7.1 that the zero vector $\vec{0}$ is not considered to be an eigenvector. However, we want E_λ to be a subspace, so it needs to contain $\vec{0}$. So, the precise description of E_λ is the set of the eigenvectors with eigenvalue λ AND $\vec{0}$.**
- Find eigenspaces (no complex eigenvalues/eigenvectors)
- What is the geometric multiplicity of an eigenvalue? (state definition)
- Know that geometric multiplicity $\leq$ algebraic multiplicity
- Know why it is bad if geometric multiplicity $\neq$ algebraic multiplicity for even one eigenvalue

Proofs to know

- **Show that the eigenspace E_λ associated to an eigenvector λ is a subspace (3 properties)**

Not eligible for the exam

- Complex eigenvalues/eigenvectors

Section 7.4

Topics to know

- What is an eigenbasis of a linear transformation T? (state definition)
- Why do we care about an eigenbasis? (Answer - the matrix of T with respect to the eigenbasis is diagonal with the eigenvalues on the diagonal. Diagonal matrices are much easier to work with than non-diagonal matrices.)

Proofs to know

- Nothing I have thought of so far. Feel free to ask.

Not eligible for the exam

- Anything past the middle of p.337
- MCMC revolution paper
- anything covered during the last 1.5 days of lecture