Homework 12.F

M27 – Linear Algebra Spring 2013 Swarthmore College Prof. Scott Cook
Assigned - April 19, 2013 Due - April 23, 2013

You may use the Intro to Markov Chains and Intro to PageRank mathematica notebooks posted on Moodle to help you on this worksheet.

Consider the game of *Yatzi*. It is like a mini version of another game (with a trademarked name) that you have probably played before. In Yatzi, a player has 3 dice. The objective is get three-of-a-kind during a turn. For each turn, a player may roll up to r times. After each roll, the player chooses which of the dice she wants to re-roll and which of the dice she wants to leave as they are. So, for example, if she rolls a pair, she will leave the pair and re-roll the third die (hoping to match the pair).

We may analyze this system using a Markov chain. After any roll, the player has either 1-of-a-kind, 2-of-a-kind, or 3-of-a-kind. Thus, we may write the player's state as a vector

$$\vec{\boldsymbol{x}} = \begin{bmatrix} \text{Probability that she has 1-of-kind} \\ \text{Probability that she has 2-of-kind} \\ \text{Probability that she has 3-of-kind} \end{bmatrix}$$

We can say that she "has" a 1-of-a-kind before the first roll, so we let $\vec{\boldsymbol{x}}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. After the first roll, there is some probability that she will still have 1-of-kind, some probability that she will now have 2-of-a-kind, and some probability that she will now have 3-of-a-kind. So, we let $\vec{\boldsymbol{x}}(1)$ be the vector whose entries are these 3 probabilities. After another roll, we have another probability distribution $\vec{\boldsymbol{x}}(2)$. And so on.

1. Find the matrix P that encodes the rules of Yatzi. (In other words, $\vec{\boldsymbol{x}}_1 = P\vec{\boldsymbol{x}}_0$, $\vec{\boldsymbol{x}}_2 = P\vec{\boldsymbol{x}}_1$, etc.)

2. Compute the probability of success (getting 3-of-a-kind) on a turn which consists of $r = 3$ rolls. (This is the number of rolls per turn allowed in the trademarked version of this game, though that game uses more dice).

3. We want to increase this probability of success by allowing the player to roll more times per turn. Find the smallest value for r that gives greater than 50% chance of success.

4. Find all of the eigenvalues of P. Observe that $|\lambda| \leq 1$ for every eigenvalue. Now, find a basis for each eigenspace. Sum the entries of each vector. What do you notice? Where possible, make each eigenvector sum to 1 by diving by the sum of its entries. Collect these vectors together into one set $\mathfrak{D}$. Is $\mathfrak{D}$ an eigenbasis of $\mathbb{R}^3$

5. Find the $\mathfrak{D}$-coordinates of $\vec{\boldsymbol{x}}_0$ and the $\mathfrak{D}$-matrix of P

6. From the expression $\vec{\boldsymbol{x}}(t) = c_1\lambda_1^t\vec{\boldsymbol{v}}_1 + c_2\lambda_2^t\vec{\boldsymbol{v}}_2 + c_3\lambda_3^t\vec{\boldsymbol{v}}_3$ from in Theorem 7.1.3, find $\lim_{r\to\infty}\vec{\boldsymbol{x}}(r)$. Thre are lots of names for this vector, such as the *limiting distribution* or the *steady state* or the *equilibrium state*, etc. We will simply call it $\vec{\boldsymbol{\pi}}$. (The entries of $\vec{\boldsymbol{\pi}}$ should sum to 1; if not, go back and fix your error). Argue that the player will certainly win, provided we allow her as many rolls as she wishes.

7. What meaning does the matrix P^2 have? How about P^3? Find $\lim_{r\to\infty} P^r$. Explain.

Now, we change our focus. In the discussion of Google PageRank in lecture, we said that PageRank dislikes websites that do not link out to any other websites. You see the reason above - the Markov chain is absorbed at those states. While this is perfectly reasonable behavior for Yatzi, it does not reflect how human surf the internet. If you come to a page with no outgoing links, you are not stuck. You simply type in some other url and re-start your surfing.

PageRank reflect this fact by tweaking P. Let q be the fraction of pages that have no outgoing links. Multiply each entry of P by $(1-q)$. Then add $\frac{q}{n}$.

8. We use the same the matrix P that you found in part 1, but now we imagine that it is being used by Google's PageRank. Modify P as discussed above. Compute $\vec{\boldsymbol{\pi}}$; the entries are the PageRanks of the websites. Note that, unlike in the Yatzi problem, no entry of $\vec{\boldsymbol{\pi}}$ is zero.