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Renormalization Group

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Motivation

- We like descriptions of a system that are scale invariant.

- 20th century physicists wanted a scale invariant theory for electrodynamics

- But they found many calculations diverged. (on small distance scales)

- At a high resolution, electron interacts with virtual particles that mess up computation.

- However! It was discovered that this high resolution setting could be described, by the same theory, it was just that the electron acted as if it had a different mass and charge.

- The process of finding the right parameters to fit a system at a particular scale is called renormalization.

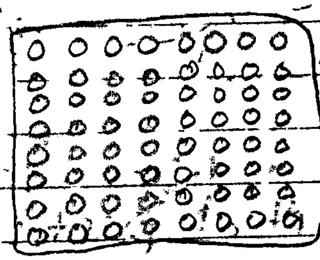
- In general, the math is pretty complicated. The renormalization group makes heavy use of functional analysis, which is the study of spaces of functions. We can iteratively apply renormalization group action to the system until it converges to a fixed point, which yields the desired parameters

- Some history: ~~Some~~ physicists (Feynman, Dyson, Dirac) did not like renormalization in the context of QED. Today it enjoys widespread support.

- This apparatus won't work for all theories. The theories which it does work are called renormalizable. Many theories (QED, QCD, electro-weak interactions) are renormalizable. Gravity is a notable example of a theory that is not renormalizable.

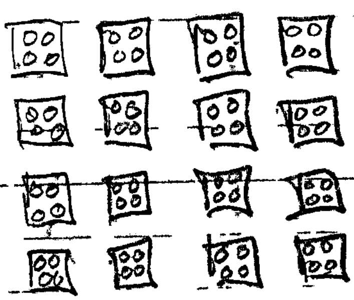
Motivating A Renormalization Approach to Stat Mech

Consider a 2D solid



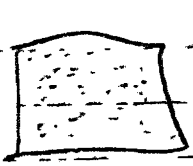
Let ~~the~~ system be given by hamiltonian H as a function of temperature T and coupling J.
I.E. $H(T, J)$

But there are so many atoms, it is hard to describe the system. Using renormalization we can think of the system as follows



∴ Only 8 atoms now but instead of T & J we use T' and J'

↓ continue



Think about the system as one big atom, with parameters T'' and J''. This is analogous to thinking of electron and associated virtual particles as a single electron with a different mass and charge.

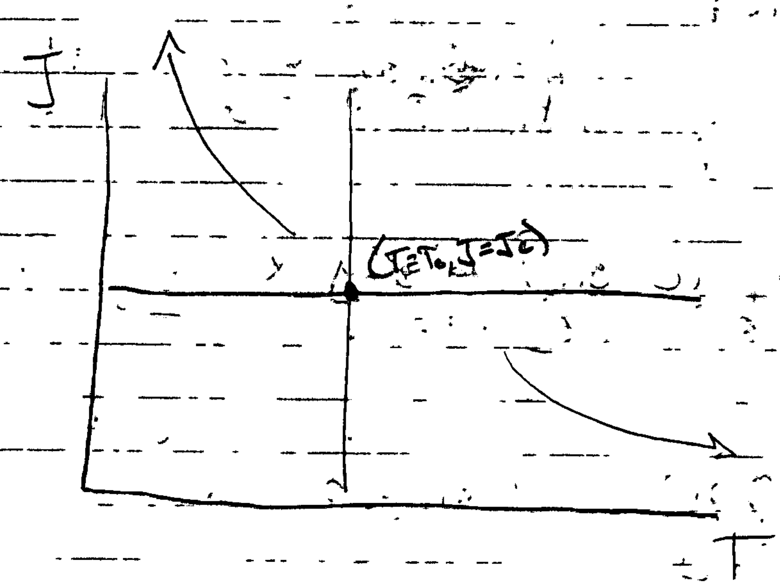
If the fixed points of the renormalization group are interesting, then we can repeatedly apply the renormalization apparatus to find the fixed points.

In the 2d Ising model (and many similar models) the fixed points are

$T=0,\ J\to\infty$ stable

$T\to\infty,\ J=0$ stable

$T=T_c,\ J=J_c$ unstable



* Show animation *

Math

The renormalization (paradoxically) is actually not a group. Instead, it is what we call a 'semigroup'. A semigroup is a tuple $(G, \cdot)$ where G is a set and $\cdot$ is a binary operator, with the properties that for $a, b \in G$, $a \cdot b \in G$, and for $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(semigroup action)

The renormalization group performs what is called 'group action' over the set of parameters that are being renormalized. Formally, semigroup action is a function $f: G \times X \to X$, where X is the set of parameters, such that for $g, h \in G$, $x \in X$, $f(g, f(h, x)) = f(g \cdot h, x)$.

Under group action f, we say x_0 is a fixed point if for all $g \in G$, $f(g, x_0) = x_0$.

The desired (renormalized) parameters are those that are fixed points under group action by the renormalization group.

In practice, we can find the fixed by stochastically composing group action. Let $c(\,)$ take a set and return a random element of the set. Let $s(\,)$ take an element ~~$x \in$~~ G and return $f(c(G), x)$. Let $s^{\circ n}(x) = \underbrace{s \circ s \circ s \circ \cdots \circ s}_{n}(x)$. Then stochastic iteration

looks like $s^{\circ N}(x)$ for $N \gg 1$. Unfortunately, this process will only find stable fixed points. Finding unstable fixed points is considerably more challenging.