Boltzmann Transport Equation

DQ

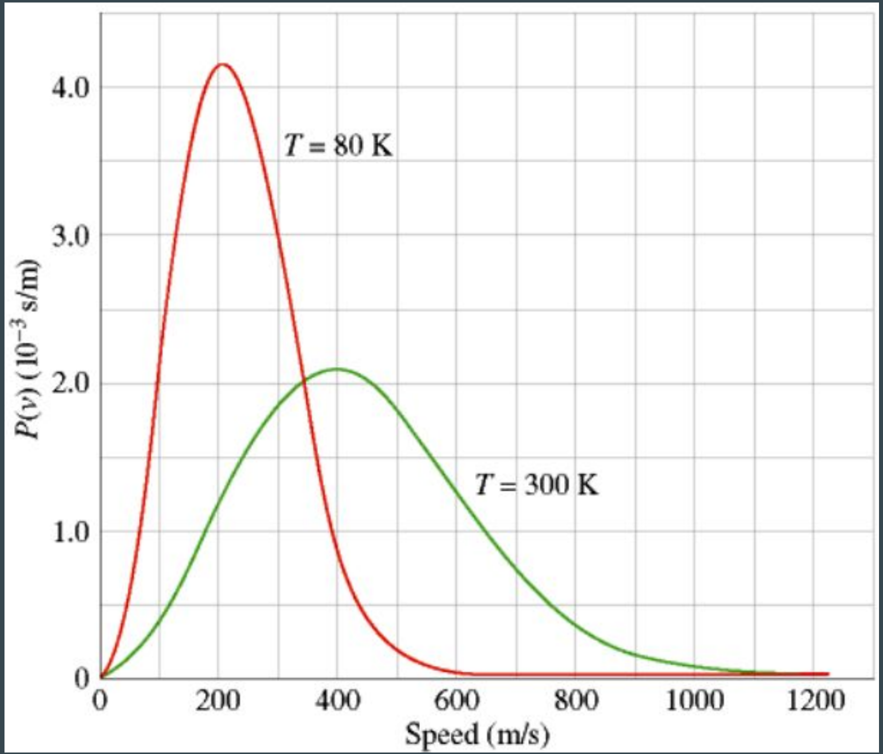
What have we learned to describe systems?

- Macroscopic

- Microscopic

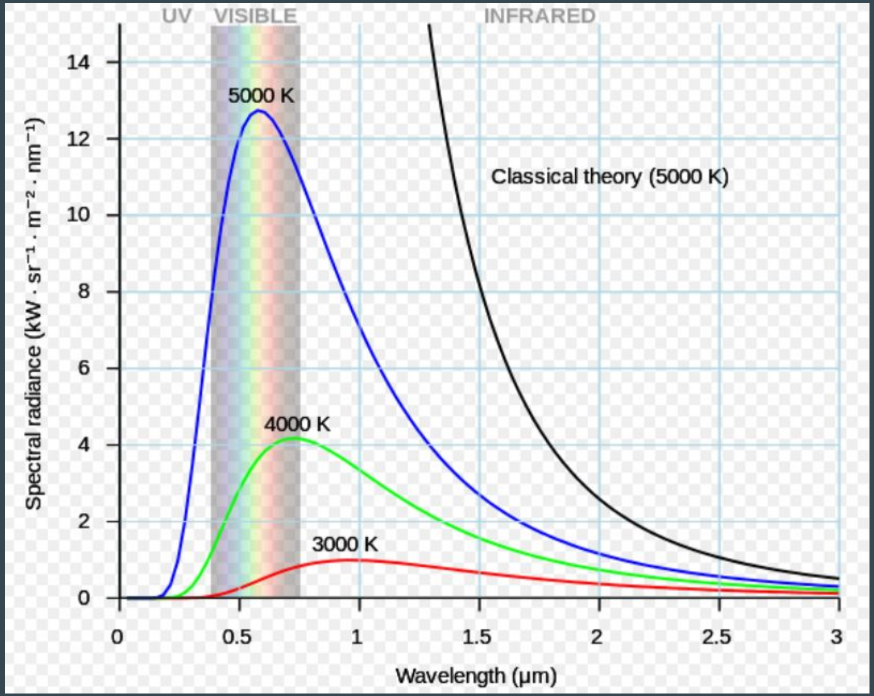
Macroscopic Systems

- Micro Canonical Ensemble
  - Fixed E, V, N
- Canonical Ensemble
  - Fixed T, V, N
  - Boltzmann Distribution of Particle Density
    - Maxwell Velocity/Speed Distribution
- Grand Canonical Ensemble
  - Fixed μ, V, T
  - Gibb’s Distribution



Microscopic Systems

- Fermions
  - Fermi Dirac Distribution
- Bosons
  - Bose Einstein Distribution
- Photons (Technically Bosons)
  - Planck Distribution



What do these systems have in common?

- A probability distribution function of some sort

$$p_i = \frac{e^{-\varepsilon_i/kT}}{\sum_{j=1}^{M} e^{-\varepsilon_j/kT}}$$

$$n_i(\varepsilon_i) = \frac{g_i}{e^{(\varepsilon_i-\mu)/kT} - 1},$$

$$\bar{\mathcal{N}}(\epsilon) = \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT} + 1}.$$

- Are there special condition to apply the distribution function?

EQUILIBRIUM!!!

Consequences of Systems Not in Equilibrium

1. Temperature becomes undefined
2. The system cannot be described with extensive or intensive quantities
3. Particle by particle treatment is needed

Boltzmann Transport Equation!

Mathematical Derivation

Step 1: Liouville's Theorem: (if no collision)

The distribution function is constant along any trajectory in phase space.

$$f\left(\vec{r}+\vec{p}\,\frac{\Delta t}{m},\ \vec{p}+\vec{F}\,\Delta t, t+\Delta t\right) \;=\; f(\vec{r},\vec{p},t)\,.$$

If there is collision, we sum it up in a "collision term".

$$f\left(\vec{r}+\vec{p}\,\frac{\Delta t}{m},\ \vec{p}+\vec{F}\,\Delta t, t+\Delta t\right) \;=\; f(\vec{r},\vec{p},t)+\left(\frac{\partial f}{\partial t}\right)_{\mathrm{Col.}}\Delta t\,.$$

Mathematical Derivation

Step 2: Take the derivative (time, space and momentum)

$$\frac{\partial f}{\partial t} = -\frac{1}{m}\vec{p}.\nabla_r f - \vec{F}.\nabla_p f + \left(\frac{\partial f}{\partial t}\right)_{\text{Col.}}$$

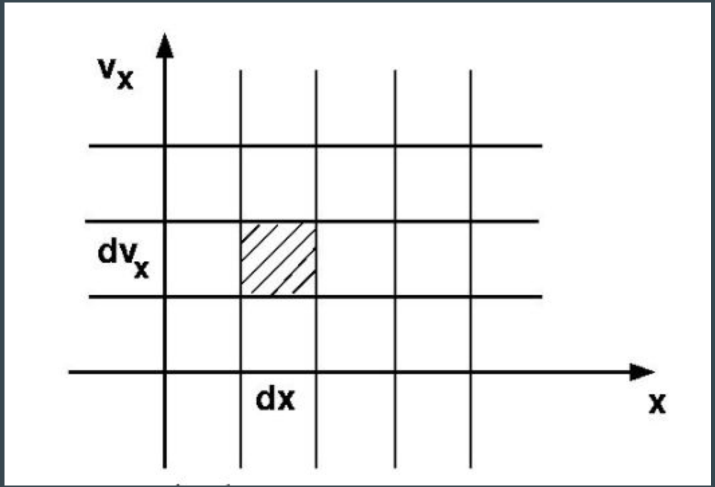
Time Deriv.

Spacial Grad.

Momentum Grad.

Intuitive Understanding (No Force at all)

Phase Space Diagram



- Distribution Function (phase space density)
  - Represented by the shadow region
- $$\frac{\partial f(x, v_x)}{\partial t} = \frac{n_{in} - n_{out}}{dt\,dx\,dv_x}$$
- assuming $v_x > 0$, $n_{in} = f(x - v_x dt, v_x)dx\,dv_x$

  $$n_{out} = f(x, v_x)dx\,dv_x$$
- $$\frac{\partial f(x, v_x)}{\partial t} = -v_x \frac{\partial f(x, v_x)}{\partial x}$$

Generalize to with External Force

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{K_x}{m} \frac{\partial f}{\partial v_x} = 0$$

Collisionless Boltzmann Transport Eq.

Kx = external force

Same?!

$$\frac{\partial f}{\partial t} = -\frac{1}{m}\vec{p}.\nabla_r f - \vec{F}.\nabla_p f + \left(\frac{\partial f}{\partial t}\right)_{\text{Col.}}$$

Collisional Boltzmann Transport Eq.

COLLISION TERM!

We can ignore the collision term... or we can talk about it

- This is crazy complicated
  - I mean trying to understand how random collisions take place in a gas...
- However, some assumptions make it easier
  - Only binary collisions (dilute gas)
  - No influence from container walls
  - No influence of the external force on the rate of collisions
  - Velocity and position of a molecule are uncorrelated (assumption of molecular chaos)

COLLISION TERM!

Rate of entering and exiting the phase space hypercube changes:

$$R(\mathrm{OUT}) = d^3p_1 f(\vec{p}_1) \int d^3p_2 \int d^3p_1' \int d^3p_2' \; f(\vec{p}_2)\, \Sigma(\vec{p}_1, \vec{p}_2, \vec{p'}_1, \vec{p'}_2).$$

$$R(\mathrm{IN}) = d^3p_1 \int d^3p_2 \int d^3p_1' \int d^3p_2' \; f(\vec{p'}_1) f(\vec{p'}_2)\, \Sigma(\vec{p}_\prime, \vec{p'}_2, \vec{p}_1, \vec{p}_2).$$

Σ denotes the rate of transition from (p1, p2) to (p1' , p2').

Collision term turns out to be:

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{Col.}} = \int d^3p_2 \int d^3p_1' \int d^3p_2' \; \Sigma(\vec{p}_1, \vec{p}_2, \vec{p'}_1, \vec{p'}_2) \Big[f(\vec{p'}_1) f(\vec{p'}_2) - f(\vec{p}_1) f(\vec{p}_2) \Big] .$$

Sanity Check

$$\frac{\partial f}{\partial t} = -\frac{1}{m}\vec{p}.\nabla_r f - \vec{F}.\nabla_p f + \left(\frac{\partial f}{\partial t}\right)_{\text{Col.}}$$

Boltzmann Transport Equation applies to non-equilibrium too, but under equilibrium, it should simplify to something familiar.

Equilibrium condition: we assume external force = 0

Solution to f: (makes the collisional term go away)

$$f(\mathbf{v}) = 4\pi v^2 \left(\frac{m}{2k\pi T}\right)^{3/2} e^{-\frac{mv^2}{2kT}}$$

Maxwell-Boltzmann Distribution!!!

Why is this useful? #1 For Plasma

Vlasov-Maxwell System of Equation:

Collisionless Boltzmann Transport Equation

$$\frac{\partial f_s}{\partial t} + \left[\mathbf{v} \cdot \nabla_r + \frac{q_s}{m_s}\left(\mathbf{E}_0 + \mathbf{v} \wedge \mathbf{B}_0\right) \cdot \nabla_v\right] f_s = 0,$$

+

Maxwell's Equations:

$$\begin{cases} \nabla \cdot \mathbf{E}_0 = \dfrac{\rho_c}{\epsilon_0} \\ \nabla \cdot \mathbf{B}_0 = 0 \\ \nabla \wedge \mathbf{E}_0 = -\dfrac{\partial \mathbf{B}_0}{\partial t} \\ \nabla \wedge \mathbf{B}_0 = \dfrac{\mathbf{j}_c}{\epsilon_0 c^2} + \dfrac{1}{c^2}\dfrac{\partial E_0}{\partial t}. \end{cases}$$

What are Vlasov Equations Good for?

Deriving MHD Equations:

Conservation of Mass, Momentum and Energy:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\
\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot \mathbf{T} &= 0 \\
\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{s} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T} &= \rho \mathbf{V}\mathbf{V} + \left(p + \frac{B^2}{8\pi}\right)\mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \\
w &= \frac{\rho V^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma - 1} \\
\mathbf{s} &= \left(\frac{\rho V^2}{2} + \frac{\gamma}{\gamma - 1} p\right)\mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{4\pi}
\end{aligned}
$$

Pressure Tensor, Energy Density, Energy Flux

Of course this process works for regular fluid too, not just magnetofluid.

#2 Regular Fluid

Conservation of mass, momentum and energy:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{u}) = \frac{\partial n}{\partial t} + \sum_i \frac{\partial}{\partial x_i}(nu_i) = 0$$

$$mn\left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j}\right) = -\sum_j \frac{\partial p_{ij}}{\partial x_j}$$

$$\frac{\partial}{\partial t}\left[mn\left(e + \frac{u^2}{2}\right)\right] + \sum_i \frac{\partial}{\partial x_i}\left[mnu_i\left(e + \frac{u^2}{2}\right) + \sum_j u_j p_{ij} + q_i\right] = 0.$$

#3 Boltzmann Equation for Annihilation (expanding universe)

"The Boltzmann equation formalizes the statement that the rate of change in the abundance of a given particle is the difference between the rate of producing and eliminating the species." (Scott Dodelson Modern Cosmology)

Let 1+2↔3+4 describe particles 1 and 2 annihilating to particles 3 and 4.

$$a^{-3}\frac{d(n_1 a^3)}{dt} = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4}$$
$$\times (2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_4)\delta(E_1 + E_2 - E_3 - E_4)|M|^2$$
$$\times \{f_3 f_4[1 \pm f_1][1 \pm f_2] - f_1 f_2[1 \pm f_3][1 \pm f_4]\}$$

Result

(0 superscript: number density of the species at equilibrium)

$$a^{-3}\frac{d\left(n_1 a^3\right)}{dt} = n_1^{(0)} n_2^{(0)} \langle\sigma v\rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}.$$

$$\langle\sigma v\rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} e^{-(E_1+E_2)/T}$$
$$\times (2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_4)\delta(E_1 + E_2 - E_3 - E_4) \left|\mathcal{M}\right|^2. \qquad (3.8)$$

Take Away Messages

1. When systems are not in equilibrium, the distribution functions we learned in Stat Mech become inapplicable.
2. It is necessary to characterize the system with the kinematics of each particle. Boltzmann Transport Equation allows you to do that.
   a. Collisionless version can be a good approximation if the gas is dilute.
   b. Collisional version is very complicated.
3. From Boltzmann Transport Equation, mass, momentum and energy conservation can be derived.
4. The application of Boltzmann Transport equation are very wide: applicable to most things that can be modeled as fluids that are not in equilibrium (or some complicated equilibrium).

http://homepage.univie.ac.at/franz.vesely/sp_english/sp/node7.html

https://arxiv.org/pdf/cond-mat/0601566.pdf

http://iopscience.iop.org/chapter/978-0-7503-1200-4/bk978-0-7503-1200-4ch1.pdf

https://www.cfa.harvard.edu/~namurphy/Lectures/Ay253_02_ConservationLaws.pdf

Dodelson Modern Cosmology