Statistical Methods in Plasma Physics

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Plasma Physics Basics

What is a Plasma?

- A system of charged particles which exhibit collective behavior

Images courtesy of Wikipedia

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Key Parameters

- Number Density, n
- Temperature, T
- Timescales, τ_p

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Describing Plasma Particle Trajectories

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- Cyclotron motion in response to constant external magnetic fields
- Drift motion in response to varied external electro-magnetic fields
- Inter-particle collisions

Describing Plasma Particle Trajectories

The Problem

For a small number of particles, we can track individual particle motion in response to both long and short range forces. However, when these particles move, they act as the source terms for Maxwell's equations. It becomes computationally impossible to solve the coupled motion equations for anything close to a real system.

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The Solution

- Simplify the model
- Approximate the results
- Iterate the process

Pros and Cons of the Phase Space Model

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Cons

- Cannot track any individual particles, only particle velocities at different specific locations
- Coupled equations featuring the distribution functions are still often intractable

Describing Distribution Function Evolution

Once we have switched to a phase space description of a plasma, we need to derive an equation which dictates the time evolution of the distribution function.

The Set Up

Imagine and infinitesimal box in phase space and track the flux out of each side.

Derivation

The total flux through the box is given by

$$\frac{df}{dt}dxdv = \begin{cases} -f(x+dx, v, t)vdv + f(x, v, t)dv) \\ -f(x, v+dv, t)a(x, v+dv, t_d x) \\ +f(x, v, t)a(x, v, t)dx \end{cases} .$$

$$\implies \frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} = \frac{\partial}{\partial v}(af).$$

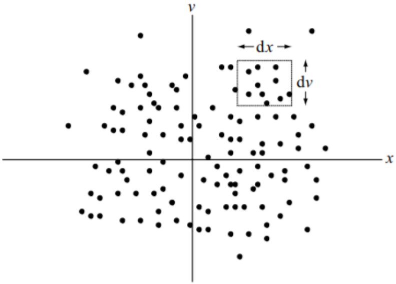

Figure: An illustration of an infinitesimal box in phase space. Courtesy of Gurnett and Bhattarcharjee, *Introduction To Plasma Physics*.

The Boltzmann Equation and the Collision operator

The Boltzmann Equation

The previous equation generalizes to

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f + \frac{\vec{F}}{m} \cdot \nabla_{\vec{v}} f = \frac{\delta_c f}{\delta t},$$

which is called the Boltzmann equation. It describes how any distribution function evolves in time.

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The Collisional Operator

The collision operator on the RHS describes collisions by invoking simultaneous particle destruction and creation in phase space. It satisfies three constraints:

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- The total number of particles is conserved during a collision
- Momentum is conserved
- Energy is conserved

The Vlasov Equation

By letting the force term in the Boltzmann equation be the Lorentz force, and setting the collision operator to zero, we obtain the Vlasov equation:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} + \frac{q}{m}[\vec{E} + \vec{v} \times \vec{B}] \cdot \nabla_{\vec{v}} f = 0.$$

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Solutions to the Vlasov Equation

This equation is still quite difficult to solve for a typical two species plasma and arbitrary external fields. But all solutions do share one quality: any distribution function which is a function of system invariants is a solution.

Equilibrium Distribution Functions

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The Maxwellian

For a two species plasma, the momentum depends on the average velocity of a species. Thus the Maxwellian is

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The Bi-Maxwellian

In strongly magnetized plasmas, the velocities are partitioned into two modes. This gives the bi-Maxwellian:

$$f = n_0 \left(\frac{m^2}{4\pi^2 k^2 T_\perp T_\parallel} \right)^{1/2} e^{-\frac{m}{2k}\left(\frac{v_\perp^2}{T_\perp} + \frac{v_\parallel^2}{T_\parallel} \right)}$$

Equations of State

These two distributions lead to wildly different plasma behavior, and which equilibrium a given plasma falls into depends on the net magnetic field inside the plasma.

MHD Equation of State

For a collisional, fully isotropized plasma, the behavior is similar to an ideal gas and follows the ideal gas EOS. Thus, adiabatic compression must conserve

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CGL Equations of State

When the velocity is partitioned into two modes, we need a different EOS for each. During adiabatic compression, these are

$$\frac{P_{\perp}}{nB} = const., \quad \frac{B^2 P_{\parallel}}{n^3} = const.$$