Riemann Zeta Function and Polylogarithm

An Introduction from a Physicist's Perspective

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Outline

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Definition

- The Riemann zeta function is defined as the *analytic continuation* of the following sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

  Notice that for real values of s, the sum only converges for $s > 1$. However, we can define the function for complex values of s by analytic continuation.
- Euler first studied this function without the knowledge of complex analysis and successfully calculated $\zeta(2)$. Riemann then extended it to complex numbers.
- It turns out that the zeta function is fundamentally related to the distribution of prime numbers.

Some Special Values

- $\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$ (Basel problem: What is the probability that two numbers selected at random are relatively prime?)
- $\zeta(4) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^4}{90}$
- $\zeta(0) = 1 + 1 + 1 + ... = -\frac{1}{2}$
- $\zeta(-1) = 1 + 2 + 3 + ... = -\frac{1}{12}$

Riemann Hypothesis

- Proposition: For all the nontrivial zeros of the zeta function, the real part of them is $\frac{1}{2}$.
- Consequences include resolution of distribution of prime numbers distribution, growth of arithmetic functions, large prime gap conjecture, etc.

Outline

Bose Integral

- We can utilize the Riemann zeta function to evaluate the **Bose integral** $I_B(n)$, which appears in the derivation for Stefan-Boltzmann law.

$$I_B(n) = \int_0^\infty dx \frac{x^n}{e^x - 1} = \int_0^\infty dx \frac{x^n e^{-x}}{1 - e^{-x}} \tag{2}$$

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- Using change of variables $y = (k+1)x$, we have

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- Thus, we have

$$I_B(n) = \zeta(n+1)\Gamma(n+1) \tag{5}$$

Stefan-Boltzamnn Constant Derivation

For a Planck spectrum (the spectrum for wavelengths ω), we are interested in the total energy calculated from the integral of that, i.e.

$$U = \int_0^\infty g(\omega) d\omega \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} = \frac{V\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3 d\omega}{e^{\beta\hbar\omega} - 1} \tag{6}$$

Using substitution $x = \hbar\beta\omega$, we have

$$U = \frac{V\hbar}{\pi^2 c^3} (\frac{1}{\hbar\beta})^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{V\hbar}{\pi^2 c^3} (\frac{1}{\hbar\beta})^4 \zeta(4)(4) \tag{7}$$

Eventually, we have

$$U = (\frac{V\pi^2 k_B^4}{15 c^3 \hbar^3}) T^4 \tag{8}$$

Thus, $u = AT^4 = U/V = (\frac{\pi^2 k_B^4}{15c^3\hbar^3}) T^4$ and $\sigma = Ac/4 = \pi^2 k_B^4 / 60 c^2 \hbar^3$.

Outline

Definition

The polylogarithm function $Li_n(z)$ is defined as

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \tag{9}$$

for the unit disk of z. Notice that we can analytically continue the function for other values of z. Also, $Li_n(1)$ is just the zeta function $\zeta(n)$.

Outline

Evaluate integrals of Bose-Einstein distribution functions

Notice that we can write

$$\frac{1}{z^{-1}e^{x}-1}=\frac{ze^{-x}}{1-ze^{-x}}=\sum_{m=0}^{\infty}(ze^{-x})^{m+1} \tag{10}$$

Then

$$\int_0^{\infty}\frac{x^{n-1}dx}{z^{-1}e^{x}-1}=\sum_{m=0}^{\infty}\int_0^{\infty}x^{n-1}(ze^{-x})^{m+1} \tag{11}$$

Proceed in similar fashion as in evaluating the Bose integral, we have (see BB C.6 for details)

$$\int_0^{\infty}\frac{x^{n-1}dx}{z^{-1}e^{x}-1}=\Gamma(n)Li_n(z) \tag{12}$$

Grand Potential of Bose-Einstein Distribution

With the polylogarithm function, we can calculate the grand potential Φ_G for non-interacting bosons as

$$\Phi_G = \frac{-2}{3}\frac{(2S+1)V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{E^{3/2}dE}{e^{\beta(E-\mu)}+1} \tag{13}$$

Writing $z = e^{\beta\mu}$, we can evaluate the grand potential as

$$\Phi_G = -k_B T\frac{(2S+1)V}{\lambda_{th}^3}Li_{5/2}(-z) \tag{14}$$

From the grand potential, we can calculate variables of our interest.

Summary

- Riemann zeta function and polylogarithm function are profound mathematically. Physicists use them to solve integrals.