Entropy

Who? Noah Rosenberg

Thermodynamic

$$\Delta S = \int \frac{\delta Q_{rev}}{T}$$

- Motivation: Carnot engine provides exact differential $\delta Q_{\mathrm{rev}}/T$, as process is reversible
- Defined by Clausius as "transformation-content" c. 1860
- From Clausius inequality: $\oint \frac{\delta Q}{T} \leq 0$

Statistical (Boltzmann) Entropy

$$S = k_B \ln \Omega$$

- Motivation: Statistical temperature (thing that is the same between two systems at equilibrium):

$$\frac{1}{k_B T} = \frac{d \ln \Omega}{dE}$$

- First law ($dU = TdS - pdV$) implies

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U} \right)_V$$

Gibbs Entropy

$$S = -k_B \sum_i P_i \ln P_i$$

- Motivation: Entropy associated with macrostates $k_B \ln \Omega$ doesn't account for energy associated with microstates, which arises out of inability to measure them–so $S_{\text{tot}} = S + S_{\text{micro}}$.
- Gibbs spoke of "Entropy as Mixedupness"
- Generalizes Boltzmann entropy by allowing different probabilities of microstates

Shannon Entropy

$$S = \langle Q \rangle = -k \sum_i Q_i P_i = -k \sum_i P_i \log P_i$$

- Motivation: information $Q = -k \log P$
- Entropy = info gained by making measurement
- If $\log \to \ln$, $k \to k_B$, then Shannon entropy is same as Gibbs

Equivalence of Definitions

- Thermodynamic and Boltzmann definitions are equivalent by construction
- Gibbs = Boltzmann under condition that all microstates have equal probability
- Gibbs = Thermodynamic in thermodynamic limit:

$$dS = -k_B \sum dp_i \ln p_i = \sum E_i dp_i / T$$

$$= \sum [d(E_i p_i) - (dE_i) p_i] / T = \frac{\delta \langle Q_{rev} \rangle}{T}$$

- Shannon = Gibbs in destruction of information (c.f. solution to Maxwell demon)

Von Neumann Entropy

$$S = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho})$$

- Motivation: Entropy associated with lack of information in quantum systems
- $\hat{\rho}$ is the density operator $\hat{\rho} = \sum P_i |\psi_i\rangle\langle\psi_i|$
- Diagonal terms in operator represent probabilities–analogous to Shannon entropy

Bekenstein-Hawking Entropy

$$S = \frac{c^3 A}{4G\hbar}$$

- Motivation:
  - Destruction of information
  - Entropic system that falls in cannot be observed–entropy vanishes!
  - Black holes are weird (perfectly understood, thus traditional ideas of entropy as "mixedupness" or lack of information are useless)
- Generalized second law of thermodynamics:

$$\Delta S_o + \Delta S_{BH} \geq 0$$

Explanation (?) of BH Entropy

- Count of microstates of matter and gravity
- Entropy of entangled QFT degrees of freedom between inside and outside of horizon
- Thermal entropy of quantum gas in atmosphere of black hole
- Conserved quantity connected with symmetry of gravitational action (Noether)
- Number of excitations of fundamental string

Moral of the Story...

- Do we understand entropy?
- Does equality of quantities imply equivalence?
- Entropy is a useful quantity in thermo/statmech–possible to calculate and manipulate; conceptualizing is much more difficult
- Third law becomes more abstract the more we learn about physics (e.g. black holes, quantum)

References

1 https://www.ohio.edu/mechanical/thermo/Intro/Chapt.1_6/Clausius/Clausius.html

2 http://www.scholarpedia.org/article/Bekenstein-Hawking_entropy

3 https://en.wikipedia.org/wiki/Entropy

4 S. Blundell and K. M. Blundell, Concepts in Thermal Physics (Oxford University Press, Oxford, 2016).

5 H. Gould and J. Tobochnik, Statistical and Thermal Physics: with Computer Applications (Princeton University Press, Princeton, N.J, 2010).

6 https://en.wikipedia.org/wiki/Gibbs_entropy