Monte Carlo Integration

PHYS114

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MC Integration is probabilistic

1. A numerical integration using random numbers
    - Rather choosing points on a grid, MC integration randomly pick points in a defined volume

2. Probabilistic in nature

    - Compare to other deterministic methods we are familiar with:
    - trapezoidal rule + Simpson’s rule

Basic mechanics of MC Integration:

$$I = \int_{\Omega} f(\overline{\mathbf{x}})\, d\overline{\mathbf{x}}$$

$$V = \int_{\Omega} d\overline{\mathbf{x}}$$

$$\overline{\mathbf{x}}_1, \cdots, \overline{\mathbf{x}}_N \in \Omega,$$

$$I \approx Q_N \equiv V \frac{1}{N} \sum_{i=1}^{N} f(\overline{\mathbf{x}}_i)$$

- By law of large #: $$\lim_{N \to \infty} Q_N = I.$$
- By Central limit Theorem : $$\sigma_N = \sqrt{\frac{\langle f^2 \rangle - \langle f \rangle^2}{N-1}}$$

Some important properties to notice

1. Powerful integration method for higher dimension
   - The error is significantly smaller compared to other deterministic integration method

$\propto 1/n$ (Midpoint rule)

$\propto 1/n^2$ (Trapezoidal rule)

$\propto 1/n^4$ (Simpson)

2. MC Integration becomes more powerful when we fine-tune the way we “randomly” pick points

   - stratified sampling
   - Importance Sampling, etc

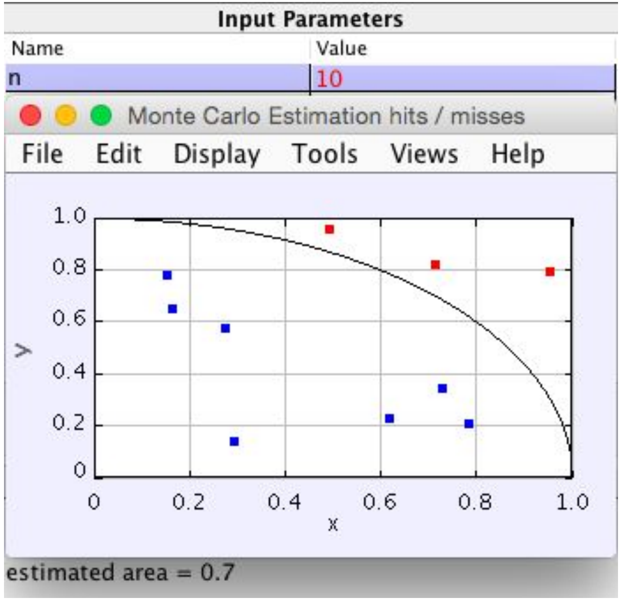
Example: G&T 3.60

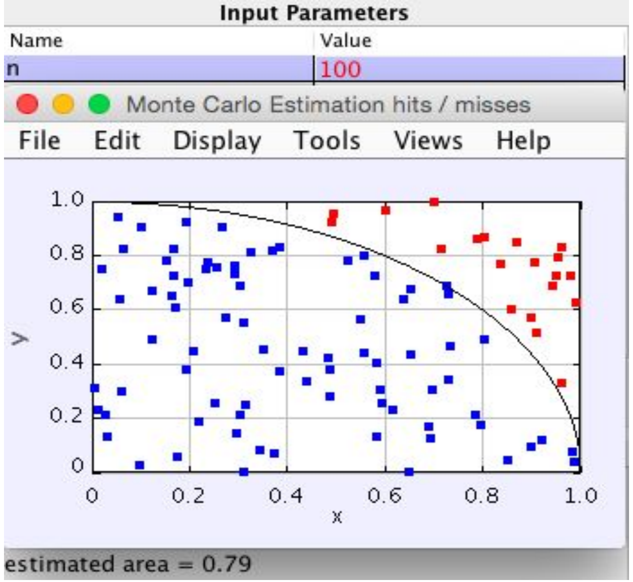
	x_i, y_i		x_i, y_i
1	0.984, 0.246	6	0.637, 0.581
2	0.860, 0.132	7	0.779, 0.218
3	0.316, 0.028	8	0.276, 0.238
4	0.523, 0.542	9	0.081, 0.484
5	0.349, 0.623	10	0.289, 0.032

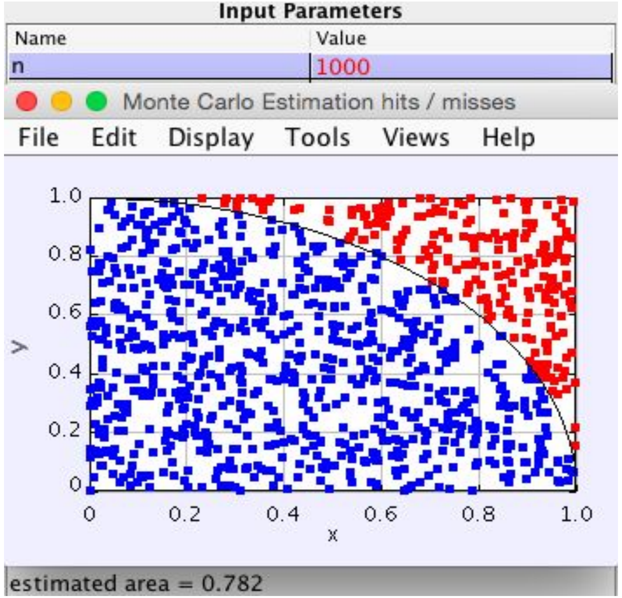
Estimate the integral using Monte Carlo Integration

$$F = \int_0^1 dx \sqrt{(1 - x^2)}.$$

Bigger N improves the MC estimation







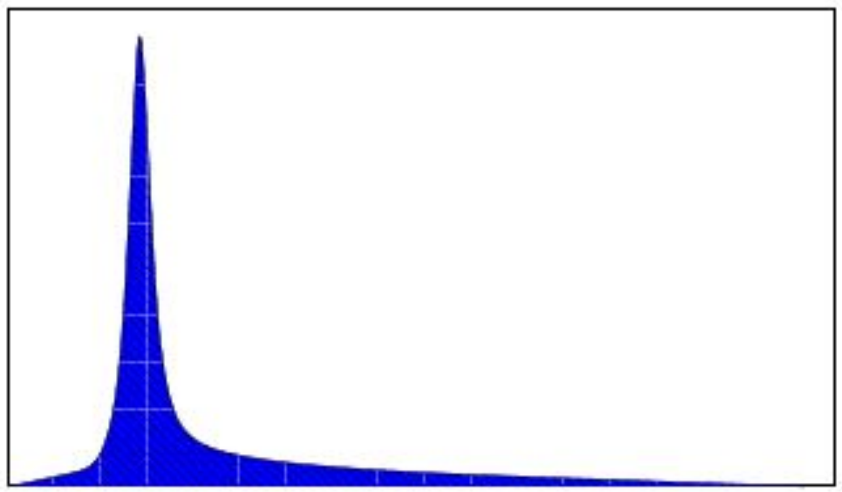
**Compare the result from MC integration to the exact result $\pi/4 \sim 0.785$

Importance sampling- A better way to sample

Core Idea

1. place higher density of points in region where integrand is large

2. Define a weight function w(x) that tells us which regions are significant



↔ Reduce σ, reduce error

Let us actualize importance sampling in math formulas...

$$I = \int dV\, p(x) \frac{f(x)}{p(x)}\,.$$

$$I = \lim_{N\to\infty} \frac{1}{N} \sum_i \frac{f(x_i)}{p(x_i)}$$

1. f/p flatter than f, σ of f/p smaller than σ of f
   → smaller error (for a given N).
2. Ideal choice: p(x) ∝ |f(x)|.

Example:

Integrate

$$I = \int_0^1 dx\,(x^{-1/3} + x/10) = 31/20 = 1.55$$

$$\sigma_N = \sqrt{\frac{\langle f^2\rangle - \langle f\rangle^2}{N-1}} \approx \frac{0.85}{\sqrt{N-1}} \qquad \sigma_N = \sqrt{\frac{\langle g^2\rangle - \langle g\rangle^2}{N-1}} \approx= \frac{0.045}{\sqrt{N-1}}$$

N	naive	importance
100	1.4878 ± 0.0751	1.5492 ± 0.0043
10000	1.5484 ± 0.0080	1.5503 ± 0.0004