Legendre Transforms

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1 Purpose of Legendre Transforms

Legendre Transforms are a method of converting a function of a set of variables into a different function in terms of conjugate variables. Conjugate variables are variables which can related to each other by the rate of change of their respective functions. For a function f(x), the variable m is a conjugate variable of x, where m is defined as:

$$m = \frac{\partial f(x)}{\partial x} = f'(x) \tag{1}$$

The slope alone cannot inform us about the values of the original function. We would need to know where the slope intercepts the y-axis in order to find f(x) for a given x value. This, in a sense, is the Legendre Transform. Described mathematically, the transformed function is the defined as the difference between the original function and the product of the conjugate variables x and m. Since we would like the transformed function to be in terms of the conjugate variable m, substitute all values of x for m.

$$g(x) = f(x) - xm \tag{2}$$

In order for Legendre Transforms to behave correctly and preserve information, the original function has to have a concavity and be monotonic over a given interval. If the function does not behave according to this for a given interval, then the Legendre Transformed function loses information of the original function. An example of a function which does not obey this is f(x) = x. Since the transform of this function is g(m) = 0, this means the reverse Legendre Transform of g(m) will not produce f(x). In other words, there's no way to preserve and extract information of the original function f(x).

Taking a geometrical approach of how to preserve information, let's consider the function f(x) = sin(x) from problem 2.32b in G&T. Since the function is not globally monotonic, we have to consider only a local domain of the function when taking the Legendre Transform. This results in a Legendre Transform of g(m) = $\sin(\cos^{-1}(m))$ - m*$\cos^{-1}(m)$. 1, the plot of g(m) against m, shows a function that only behaves in the domain of -1 to 1. Since sin(x) is not monotonic everywhere, the reverse Legendre Transform won't preserve information for all values of x.

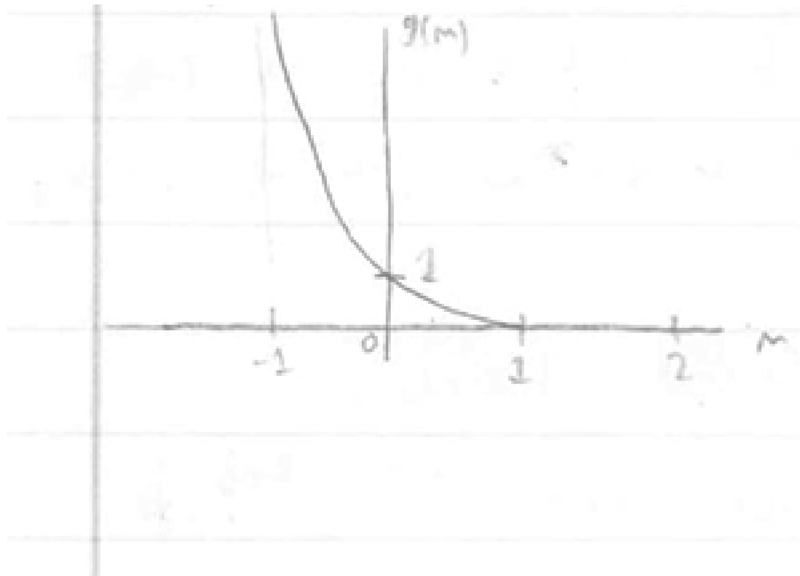

Figure 1: Legendre Transform of f(x) = sin(x). The transform is g(m) = sin($\cos^{-1}$(m)) - m*$\cos^{-1}$(m)

2 Legendre Transforms in Physics

2.1 Thermodynamics

There are many instances in thermodynamics where we are not able to understand the properties of a system due to not being able to measure the system's state variables. As such, we have to take the Legendre Transform of other state functions in the system to extract the information we desire.

As an example let's imagine a box, containing some gas, is left in a heat bath with known temperature and pressure. We would like to find the internal energy, U, of the box. However, knowing U requires that we know both the entropy and volume of the system, which may be difficult to measure for the box. However, we can use the concept of free energy from the work the heat bath does to the box to understand internal energy U. By knowing the temperature, T, and pressure, P, of the heat bath we can know the Gibbs Free Energy, which can be used to find U. The reason why we are able to know the state functions of one system to understand the state functions of another is because each of the energies and free energies are Legendre Transforms of each other. The conjugate

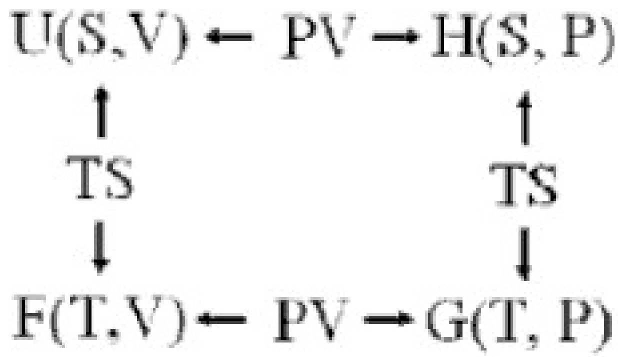

Figure 2: A flow chart showing the relationship between the energies and free energies of a given system. Source: http://www.eoht.info/page/Legendre+transform

variables of this thermodynamical system are the pairs S & T, and V & P. As such, we can find U of the box without having to measure entropy, S, or volume, V.

Interestingly, the product of conjugate variables share the same units as their respective functions because they are the work that's being performed.

2.2 Lagrangian and Hamiltonian Mechanics

Legendre Transforms are also applicable when switching from Lagrangian Mechanics to Hamiltonian Mechanics, and vice versa. Velocity, v, and momentum, p, are conjugate variables for L(x,v) and H(x,p). Although in classical mechanics v and p are related by p = mv, switching between systems can be useful when in quantum mechanics we know the wavelength of a particle but not its velocity.