Legendre Transforms

What's the point?

Thermodynamic potential is an important concept in statistical physics. Unfortunately, U is a function of S, which is difficult to control in lab settings. So we have created an assortment of other tools to use:

Internal Energy: $U(S, V, N)$
Entropy : $S(U, V, N)$
Helmholtz : $F(T, V, N)$
Enthalpy : $H(S, P, N)$
Gibbs : $G(T, P, N)$
Landau : $\Omega(T, V, \mu)$

We have also seen the need to have different language for the same idea in classical physics, with the lagrangian and hamiltonian. In classical physics, the lagrangian generally seems easier to work with, but in quantum physics, we almost exclusively used the hamiltonian.

So, we see that its nice to have different ways of thinking about a concept. How does that relate to Legendre transforms?

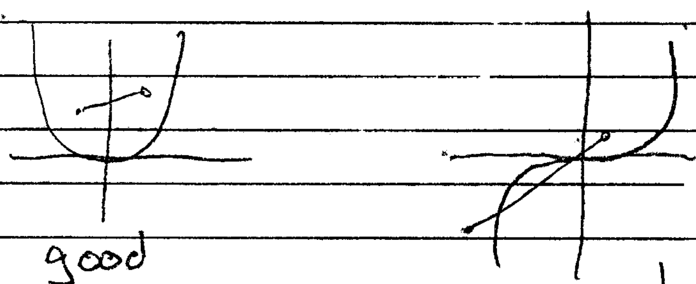
Legendre transforms are the means by which we can navigate between these different formulations. Let's examine explicitely how it works.

First, I'm going to introduce some terminology.

Def Convex Function:

$$\forall x_1, x_2 \in X,\ \forall t \in [0,1],\ f(tx_1 + (1-t)x_2) \leq t f(x_1) + (1-t) f(x_2)$$

This definition can make the concept seem hard to grasp, so we can think about convex functions more intuively as follows. Give some graph $f(x)$, f is convex iff for any two pts I pick above the graph, I can join them with a line without crossing the graph.



good

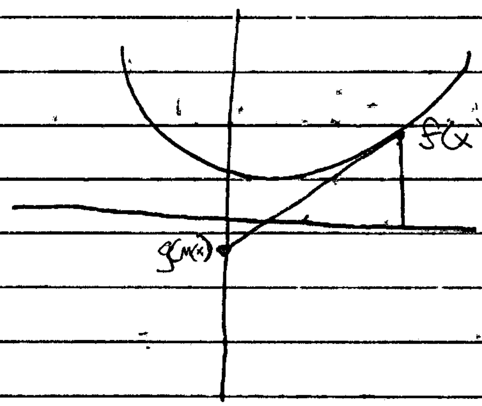
not good

To define a Legendre transform on a function, it suffices for the function to be convex.

Now, in physics we usually assume that our functions are smooth, i.e. that it is infinitely differentiable everywhere in the domain. So lets think about what convex means in this context. The way I find easiest is that f' must be a strictly monotonically increasing function of x. Graphically if I draw a horizontal line thru the graph, it should only pass thru once.

Then given our slope m, we define the Legendre transform

$$G(m) = m \cdot x(m) - f(x(m))$$



graphical representation

Cool Dualities

$G(m) + f(x) = mx$

$f_{min} = -G(0)$!!!

$\frac{d^2G}{dm^2} \cdot \frac{d^2f}{dx^2} = 1$ or $g' \circ f'(x) = x$, $f' \circ g'(m) = m$

tools to check that you did the transformation right

Returning to a physics perspective how will this work?

Say we have $f(x_1, \dots, x_n)$ but maybe some of the latter variables are difficult to measure.

$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)_{x_{j \neq i}} dx_i$

Now we have some new variables $u_i = \left(\frac{\partial f}{\partial x_i}\right)_{x_{j \neq i}}$

Now let us define $g = f - \sum_{p+1}^{n} u_i x_i$

So $dg = df - \sum_{i=p+1}^{n} \left[u_i dx_i + x_i du_i \right]$

$$= \sum_{i=1}^{n} u_i dx_i - \sum_{i=p+1}^{n} u_i dx_i - \sum_{i=p+1}^{n} x_i du_i$$

$$= \sum_{i=1}^{p} u_i dx_i + \sum_{i=p+1}^{n} (-x_i) du_i$$

Now $g = g(x_1, \dots, x_p, u_{p+1}, \dots, u_n)$ is a function of different variables, i.e. the Legendre transform.

Remember $g = \mathcal{F} - \sum_{i=p+1}^{n} \left(\frac{\partial \mathcal{F}}{\partial x_i}\right)_{x_{j \neq i}} X_i$

Say $E = E(S, V, N)$ and we want to remove our entropy dependence

Write $g = E - \left(\frac{\partial E}{\partial S}\right)_{V,N} \cdot S$

$dE = TdS - pdV + \mu dN$ so $\left(\frac{\partial E}{\partial S}\right)_{V,N} = T$

So $g = E - TS$

This is just Helmholtz free energy. Now lets make sure this actually did what we wanted it to. IE is g a function of S?

$$F = E - TS$$

$$\begin{aligned} dF &= dE - dTS - TdS \\ &= TdS - pdV + \mu dN - dTS - TdS \\ &= -pdV + \mu dN - SdT \end{aligned}$$

$$F = F(V, N, T)$$