7.6 Bose-Einstein Condensation

The previous two sections treated bosons (photons and phonons) that can be created in arbitrary numbers—whose total number is determined by the condition of thermal equilibrium. But what about more "ordinary" bosons, such as atoms with integer spin, whose number is fixed from the outset?

I've saved this case for last because it is more difficult. In order to apply the Bose-Einstein distribution we'll have to determine the chemical potential, which (rather than being fixed at zero) is now a nontrivial function of the density and temperature. Determining μ will require some careful analysis, but is worth the trouble: We'll find that it behaves in a most peculiar way, indicating that a gas of bosons will abruptly "condense" into the ground state as the temperature goes below a certain critical value.

It's simplest to first consider the limit $T \to 0$. At zero temperature, all the atoms will be in the lowest-energy available state, and since arbitrarily many bosons are allowed in any given state, this means that *every* atom will be in the ground state. (Here again, when I say simply "state" I mean a single-particle state.) For atoms confined to a box of volume $V = L^3$, the energy of the ground state is

$$\epsilon_0 = \frac{h^2}{8mL^2}(1^2+1^2+1^2) = \frac{3h^2}{8mL^2}, \tag{7.118}$$

which works out to a *very* small energy provided that L is macroscopic. At any temperature, the average number of atoms in this state, which I'll call N_0, is given by the Bose-Einstein distribution:

$$N_0 = \frac{1}{e^{(\epsilon_0-\mu)/kT}-1}. \tag{7.119}$$

When T is sufficiently low, N_0 will be quite large. In this case, the denominator of this expression must be very small, which implies that the exponential is very close to 1, which implies that the exponent, $(\epsilon_0-\mu)/kT$, is very small. We can therefore expand the exponential in a Taylor series and keep only the first two terms, to obtain

$$N_0 = \frac{1}{1+(\epsilon_0-\mu)/kT-1} = \frac{kT}{\epsilon_0-\mu} \qquad \text{(when } N_0 \gg 1\text{)}. \tag{7.120}$$

The chemical potential μ, therefore, must be equal to ϵ_0 at $T=0$, and just a *tiny* bit less than ϵ_0 when T is nonzero but still sufficiently small that nearly all of the atoms are in the ground state. The remaining question is this: How low must the temperature be, in order for N_0 to be large?

The general condition that determines μ is that the sum of the Bose-Einstein distribution over *all* states must add up to the total number of atoms, N:

$$N = \sum_{\text{all } s} \frac{1}{e^{(\epsilon_s-\mu)/kT}-1}. \tag{7.121}$$

In principle, we could keep guessing values of μ until this sum works out correctly (and repeat the process for each value of T). In practice, it's usually easier to convert the sum to an integral:

$$N = \int_0^\infty g(\epsilon) \frac{1}{e^{(\epsilon-\mu)/kT} - 1}\, d\epsilon. \tag{7.122}$$

This approximation should be valid when $kT \gg \epsilon_0$, so that the number of terms that contribute significantly to the sum is large. The function $g(\epsilon)$ is the **density of states**: the number of single-particle states per unit energy. For spin-zero bosons confined in a box of volume V, this function is the same as what we used for electrons in Section 7.3 (equation 7.51) but divided by 2 because now there is only one spin orientation:

$$g(\epsilon) = \frac{2}{\sqrt{\pi}} \Big(\frac{2\pi m}{h^2}\Big)^{3/2} V\sqrt{\epsilon}. \tag{7.123}$$

Figure 7.31 shows graphs of the density of states, the Bose-Einstein distribution (drawn for μ slightly less than zero), and the product of the two, which is the distribution of particles as a function of energy.

Unfortunately, the integral 7.122 cannot be performed analytically. Therefore we must guess values of μ until we find one that works, doing the integral numerically each time. The most interesting (and easiest) guess is $\mu = 0$, which should work (to a good approximation) at temperatures that are low enough for N_0 to be large. Plugging in $\mu = 0$ and changing variables to $x = \epsilon/kT$ gives

$$\begin{aligned} N &= \frac{2}{\sqrt{\pi}} \Big(\frac{2\pi m}{h^2}\Big)^{3/2} V \int_0^\infty \frac{\sqrt{\epsilon}\, d\epsilon}{e^{\epsilon/kT} - 1} \\ &= \frac{2}{\sqrt{\pi}} \Big(\frac{2\pi mkT}{h^2}\Big)^{3/2} V \int_0^\infty \frac{\sqrt{x}\, dx}{e^x - 1}. \end{aligned} \tag{7.124}$$

The integral over x is equal to 2.315; combining this number with the factor of $2/\sqrt{\pi}$ yields the formula

$$N = 2.612 \Big(\frac{2\pi mkT}{h^2}\Big)^{3/2} V. \tag{7.125}$$

This result is obviously wrong: Everything on the right-hand side is independent

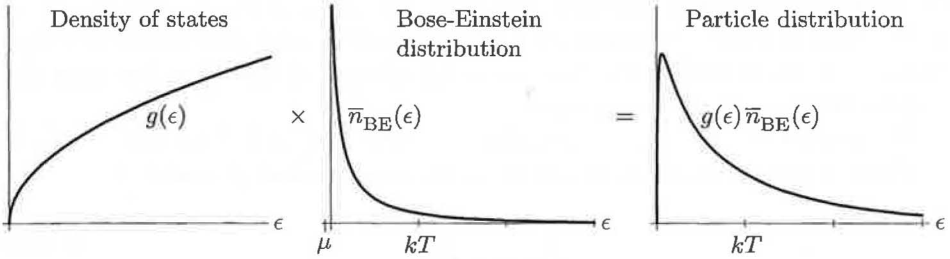

Figure 7.31. The distribution of bosons as a function of energy is the product of two functions, the density of states and the Bose-Einstein distribution.

of temperature except T, so it says that the number of atoms depends on the temperature, which is absurd. In fact, there can be only one particular temperature for which equation 7.125 is correct; I'll call this temperature T_c:

$$N = 2.612\left(\frac{2\pi mkT_c}{h^2}\right)^{3/2} V, \qquad \text{or} \qquad kT_c = 0.527\left(\frac{h^2}{2\pi m}\right)\left(\frac{N}{V}\right)^{2/3}. \tag{7.126}$$

But what's wrong with equation 7.125 when $T \neq T_c$? At temperatures *higher* than T_c, the chemical potential must be significantly less than zero; from equation 7.122 you can see that a negative value of μ will yield a result for N that is smaller than the right-hand side of equation 7.125, as desired. At temperatures *lower* than T_c, on the other hand, the solution to the paradox is more subtle; in this case, replacing the discrete sum 7.121 with the integral 7.122 is invalid.

Look carefully at the integrand in equation 7.124. As ϵ goes to zero, the density of states (proportional to $\sqrt{\epsilon}$) goes to zero while the Bose-Einstein distribution blows up (in proportion to $1/\epsilon$). Although the product is an integrable function, it is not at all clear that this infinite spike at $\epsilon = 0$ correctly represents the sum 7.121 over the actual discretely spaced states. In fact, we have already seen in equation 7.120 that the number of atoms in the ground state can be enormous when $\mu \approx 0$, and this enormous number is not included in our integral. On the other hand, the integral *should* correctly represent the number of particles in the vast majority of the states, away from the spike, where $\epsilon \gg \epsilon_0$. If we imagine cutting off the integral at a lower limit that is somewhat greater than ϵ_0 but much less than kT, we'll still obtain *approximately* the same answer,

$$N_{\text{excited}} = 2.612\left(\frac{2\pi mkT}{h^2}\right)^{3/2} V \qquad (\text{when } T < T_c). \tag{7.127}$$

This is then the number of atoms in excited states, *not* including the ground state. (Whether this expression correctly accounts for the few *lowest* excited states, just above the ground state in energy, is not completely clear. If we assume that the difference between N and the preceding expression for N_{excited} is sufficiently large, then it follows that μ must be much closer to the ground-state energy than to the energy of the first excited state, and therefore that no excited state contains anywhere near as many atoms as the ground state. However, there will be a narrow range of temperatures, just below T_c, where this condition is not met. When the total number of atoms is not particularly large, this range of temperatures might not even be so narrow. These issues are explored in Problem 7.66.)

So the bottom line is this: At temperatures higher than T_c, the chemical potential is negative and essentially all of the atoms are in excited states. At temperatures lower than T_c, the chemical potential is very close to zero and the number of atoms in excited states is given by equation 7.127; this formula can be rewritten more simply as

$$N_{\text{excited}} = \left(\frac{T}{T_c}\right)^{3/2} N \qquad (T < T_c). \tag{7.128}$$

The rest of the atoms must be in the ground state, so

$$N_0 = N - N_{\text{excited}} = \left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] N \qquad (T < T_c). \tag{7.129}$$

Figure 7.32 shows a graph of N_0 and N_{excited} as functions of temperature; Figure 7.33 shows the temperature dependence of the chemical potential.

The abrupt accumulation of atoms in the ground state at temperatures below T_c is called **Bose-Einstein condensation**. The transition temperature T_c is called the **condensation temperature**, while the ground-state atoms themselves are called the **condensate**. Notice from equation 7.126 that the condensation temperature is (aside from the factor of 2.612) precisely the temperature at which the quantum volume ($v_Q = (h^2/2\pi mkT)^{3/2}$) equals the average volume per particle (V/N). In other words, if we imagine the atoms being in wavefunctions that are as localized in space as possible (as in Figure 7.4), then condensation begins to occur

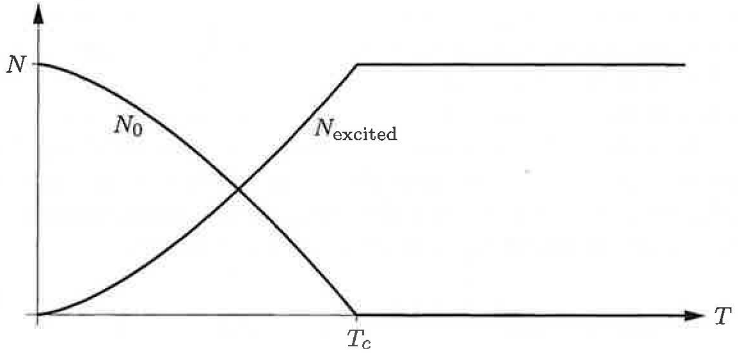

Figure 7.32. Number of atoms in the ground state (N_0) and in excited states, for an ideal Bose gas in a three-dimensional box. Below T_c the number of atoms in excited states is proportional to $T^{3/2}$.

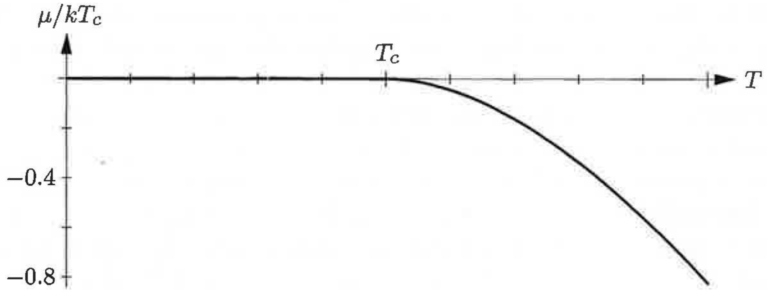

Figure 7.33. Chemical potential of an ideal Bose gas in a three-dimensional box. Below the condensation temperature, μ differs from zero by an amount that is too small to show on this scale. Above the condensation temperature μ becomes negative; the values plotted here were calculated numerically as described in Problem 7.69.

just as the wavefunctions begin to overlap significantly. (The condensate atoms themselves have wavefunctions that occupy the entire container, which I won't try to draw.)

Numerically, the condensation temperature turns out to be very small in all realistic experimental situations. However, it's not as low as we might have guessed. If you put a *single* particle into a box of volume V, it's reasonably likely to be found in the ground state only when kT is of order ϵ_0 or smaller (so that the excited states, which have energies of $2\epsilon_0$ and higher, are significantly less probable). However, if you put a *large* number of identical bosons into the same box, you can get most of them into the ground state at temperatures only somewhat less than T_c, which is much higher: From equations 7.118 and 7.126 we see that kT_c is greater than ϵ_0 by a factor of order $N^{2/3}$. The hierarchy of energy scales—$(\epsilon_0 - \mu) \ll \epsilon_0 \ll kT_c$—is depicted schematically in Figure 7.34.

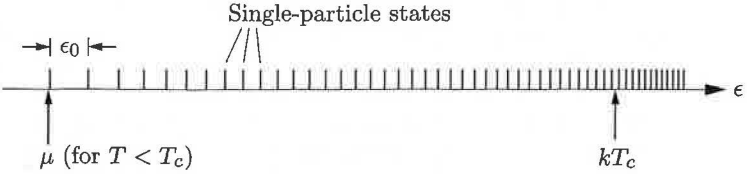

Figure 7.34. Schematic representation of the energy scales involved in Bose-Einstein condensation. The short vertical lines mark the energies of various single-particle states. (Aside from growing closer together (on average) with increasing energy, the locations of these lines are not quantitatively accurate.) The condensation temperature (times k) is many times larger than the spacing between the lowest energy levels, while the chemical potential, when $T < T_c$, is only a tiny amount below the ground-state energy.

Real-World Examples

Bose-Einstein condensation of a gas of weakly interacting atoms was first achieved in 1995, using rubidium-87.* In this experiment, roughly 10^4 atoms were confined (using the laser cooling and trapping technique described in Section 4.4) in a volume of order 10^{-15} m^3. A large fraction of the atoms were observed to condense into the ground state at a temperature of about 10^{-7} K, a hundred times greater than the temperature at which a *single* isolated atom would have a good chance of being in the ground state. Figure 7.35 shows the velocity distribution of the atoms in this experiment, at temperatures above, just below, and far below the condensation temperature. As of 1999, Bose-Einstein condensation has also been achieved with dilute gases of atomic sodium, lithium, and hydrogen.

---

*For a beautiful description of this experiment see Carl E. Wieman, "The Richtmyer Memorial Lecture: Bose-Einstein Condensation in an Ultracold Gas," *American Journal of Physics* **64**, 847–855 (1996).

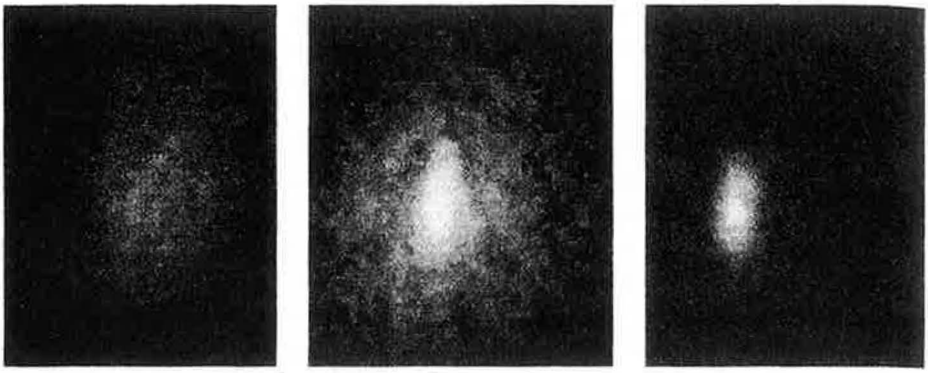
$T = 200$ nK $T = 100$ nK $T \approx 0$

Figure 7.35. Evidence for Bose-Einstein condensation of rubidium-87 atoms. These images were made by turning off the magnetic field that confined the atoms, letting the gas expand for a moment, and then shining light on the expanded cloud to map its distribution. Thus, the positions of the atoms in these images give a measure of their *velocities* just before the field was turned off. Above the condensation temperature (left), the velocity distribution is broad and isotropic, in accord with the Maxwell-Boltzmann distribution. Below the condensation temperature (center), a substantial fraction of the atoms fall into a small, elongated region in velocity space. These atoms make up the condensate; the elongation occurs because the trap is narrower in the vertical direction, causing the ground-state wavefunction to be narrower in position space and thus wider in velocity space. At the lowest temperatures achieved (right), essentially all of the atoms are in the ground-state wavefunction. From Carl E. Wieman, *American Journal of Physics* **64**, 854 (1996).

Bose-Einstein condensation also occurs in systems where particle interactions are significant, so that the quantitative treatment of this section is not very accurate. The most famous example is liquid helium-4, which forms a **superfluid** phase, with essentially zero viscosity, at temperatures below 2.17 K (see Figure 5.13). More precisely, the liquid below this temperature is a mixture of normal and superfluid components, with the superfluid becoming more predominant as the temperature decreases. This behavior suggests that the superfluid component is a Bose-Einstein condensate; indeed, a naive calculation, ignoring interatomic forces, predicts a condensation temperature only slightly greater than the observed value (see Problem 7.68). Unfortunately, the superfluid property itself cannot be understood without accounting for interactions between the helium atoms.

If the superfluid component of helium-4 is a Bose-Einstein condensate, then you would think that helium-3, which is a fermion, would have no such phase. And indeed, it has no superfluid transition anywhere near 2 K. Below 3 *milli*kelvin, however, ^{3}He turns out to have not one but *two* distinct superfluid phases.* How

---

*These phases were discovered in the early 1970s. To achieve such low temperatures the experimenters used a helium dilution refrigerator (see Section 4.4) in combination with the cooling technique described in Problem 5.34.

is this possible for a system of fermions? It turns out that the "particles" that condense are actually *pairs* of ^{3}He atoms, held together by the interaction of their nuclear magnetic moments with the surrounding atoms.* A pair of fermions has integer spin and is therefore a boson. An analogous phenomenon occurs in a superconductor, where pairs of electrons are held together through interactions with the vibrating lattice of ions. At low temperature these pairs "condense" into a superconducting state, yet another example of Bose-Einstein condensation.†

Why Does it Happen?

Now that I've shown you that Bose-Einstein condensation *does* happen, let me return to the question of *why* it happens. The derivation above was based entirely on the Bose-Einstein distribution function—a powerful tool, but not terribly intuitive. It's not hard, though, to gain some understanding of this phenomenon using more elementary methods.

Suppose that, instead of a collection of identical bosons, we have a collection of N *distinguishable* particles all confined inside a box. (Perhaps they're all painted different colors or something.) Then, if the particles don't interact with each other, we can treat each one of them as a separate system using Boltzmann statistics. At temperature T, a given particle has a decent chance of occupying any single-particle state whose energy is of order kT, and the number of such states will be quite large under any realistic conditions. (This number is essentially equal to the single-particle partition function, Z_1.) The probability of the particle being in the ground state is therefore very small, namely $1/Z_1$. Since this conclusion applies separately to each one of the N distinguishable particles, only a tiny fraction of the particles will be found in the ground state. There is no Bose-Einstein condensation.

It's useful to analyze this same situation from a different perspective, treating the entire system all at once, rather than one particle at a time. From this viewpoint, each *system* state has its own probability and its own Boltzmann factor. The system state with all the particles in the ground state has a Boltzmann factor of 1 (taking the ground-state energy to be zero for simplicity), while a system state with total energy U has a Boltzmann factor of $e^{-U/kT}$. According to the conclusion of the previous paragraph, the dominant system states are those for which nearly all of the particles are in excited states with energies of order kT; the total system energy is therefore $U \sim NkT$, so the Boltzmann factor of a typical system state is something like $e^{-NkT/kT} = e^{-N}$. This is a *very* small number! How can it be that the system prefers these states, rather than condensing into the ground state with its much larger Boltzmann factor?

The answer is that while any *particular* system state with energy of order NkT is highly improbable, the *number* of such states is so huge that taken together they

---

*For an overview of the physics of both isotopes of liquid helium, see Wilks and Betts (1987).

†For review articles on Bose-Einstein condensation in a variety of systems, see A. Griffin, D. W. Snoke, and S. Stringari, eds., *Bose-Einstein Condensation* (Cambridge University Press, Cambridge, 1995).

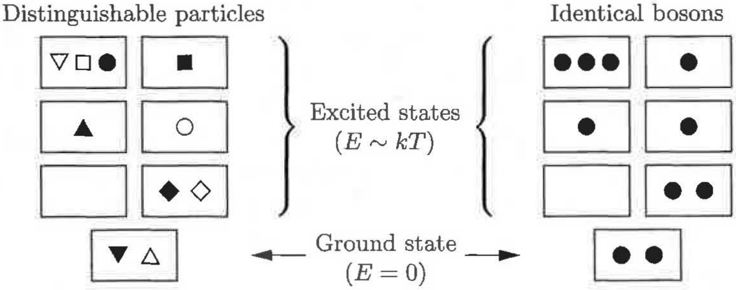

Figure 7.36. When most particles are in excited states, the Boltzmann factor for the entire system is always very small (of order e^{-N}). For distinguishable particles, the number of arrangements among these states is so large that system states of this type are still very probable. For identical bosons, however, the number of arrangements is much smaller.

are quite probable after all (see Figure 7.36). The number of ways of arranging N distinguishable particles among Z_1 single-particle states is Z_1^N, which overwhelms the Boltzmann factor e^{-N} provided that $Z_1 \gg 1$.

Now let's return to the case of identical bosons. Here again, if essentially all the particles are in single-particle states with energies of order kT, then the system state has a Boltzmann factor of order e^{-N}. But now, the *number* of such system states is much smaller. This number is essentially the number of ways of arranging N indistinguishable particles among Z_1 single-particle states, which is mathematically the same as the number of ways of arranging N units of energy among Z_1 oscillators in an Einstein solid:

$$\begin{pmatrix}\text{number of}\\ \text{system states}\end{pmatrix} \sim \binom{N+Z_1-1}{N} \sim \begin{cases}(eZ_1/N)^N & \text{when } Z_1 \gg N;\\ (eN/Z_1)^{Z_1} & \text{when } Z_1 \ll N.\end{cases} \tag{7.130}$$

When the number of available single-particle states is much larger than the number of bosons, the combinatoric factor is again large enough to overwhelm the Boltzmann factor e^{-N}, so system states with essentially all the bosons in excited states will again predominate. On the other hand, when the number of available single-particle states is much smaller than the number of bosons, the combinatoric factor is not large enough to compensate for the Boltzmann factor, so these system states, even all taken together, will be exponentially improbable. (This last conclusion is not quite clear from looking at the formulas, but here is a simple numerical example: When $N = 100$ and $Z_1 = 25$, a system state with all the bosons in excited states has a Boltzmann factor of order $e^{-100} = 4 \times 10^{-44}$, while the number of such system states is only $\binom{124}{100} = 3 \times 10^{25}$.) In general, the combinatoric factor will be sufficiently large to get about one boson, on average, into each available excited state. Any remaining bosons condense into the ground state, because of the way the Boltzmann factor favors system states with lower energy.

So the explanation of Bose-Einstein condensation lies in the combinatorics of counting arrangements of identical particles: Since the number of distinct ways of arranging identical particles among the excited states is relatively small, the ground state becomes much more favored than if the particles were distinguishable. You may still be wondering, though, how we *know* that bosons of a given species are truly identical and must therefore be counted in this way. Or alternatively, how do we *know* that the fundamental assumption, which gives all distinct states (of the system plus its environment) the same statistical weight, applies to systems of identical bosons? These questions have good theoretical answers, but the answers require an understanding of quantum mechanics that is beyond the scope of this book. Even then, the answers are not completely airtight—there is still the possibility that *some* undiscovered type of interaction may be able to distinguish supposedly identical bosons from each other, causing a Bose-Einstein condensate to spontaneously evaporate. So far, the experimental fact is that such interactions do not seem to exist. Let us therefore invoke Occam's Razor and conclude, if only tentatively, that bosons of a given species are truly indistinguishable; as David Griffiths has said,* even God cannot tell them apart.

Problem 7.65. Evaluate the integral in equation 7.124 numerically, to confirm the value quoted in the text.

Problem 7.66. Consider a collection of 10,000 atoms of rubidium-87, confined inside a box of volume $(10^{-5}\text{ m})^3$.

- (a) Calculate ϵ_0, the energy of the ground state. (Express your answer in both joules and electron-volts.)
- (b) Calculate the condensation temperature, and compare kT_c to ϵ_0.
- (c) Suppose that $T = 0.9T_c$. How many atoms are in the ground state? How close is the chemical potential to the ground-state energy? How many atoms are in each of the (threefold-degenerate) first excited states?
- (d) Repeat parts (b) and (c) for the case of 10^6 atoms, confined to the same volume. Discuss the conditions under which the number of atoms in the ground state will be much greater than the number in the first excited state.

Problem 7.67. In the first achievement of Bose-Einstein condensation with atomic hydrogen,† a gas of approximately 2×10^{10} atoms was trapped and cooled until its peak density was 1.8×10^{14} atoms/cm^3. Calculate the condensation temperature for this system, and compare to the measured value of 50 μK.

Problem 7.68. Calculate the condensation temperature for liquid helium-4, pretending that the liquid is a gas of noninteracting atoms. Compare to the observed temperature of the superfluid transition, 2.17 K. (The density of liquid helium-4 is 0.145 g/cm^3.)

Problem 7.69. If you have a computer system that can do numerical integrals, it's not particularly difficult to evaluate μ for $T > T_c$.

- (a) As usual when solving a problem on a computer, it's best to start by putting everything in terms of dimensionless variables. So define $t = T/T_c$,

---

[Introduction to Quantum Mechanics (Prentice-Hall, Englewood Cliffs, NJ, 1995), page 179.

†Dale G. Fried et al., *Physical Review Letters* **81**, 3811 (1998).

$c = \mu/kT_c$, and $x = \epsilon/kT_c$. Express the integral that defines μ, equation 7.122, in terms of these variables. You should obtain the equation

$$2.315 = \int_0^\infty \frac{\sqrt{x}\,dx}{e^{(x-c)/t} - 1}.$$

(b) According to Figure 7.33, the correct value of c when $T = 2T_c$ is approximately -0.8. Plug in these values and check that the equation above is approximately satisfied.

(c) Now vary μ, holding T fixed, to find the precise value of μ for $T = 2T_c$. Repeat for values of T/T_c ranging from 1.2 up to 3.0, in increments of 0.2. Plot a graph of μ as a function of temperature.

Problem 7.70. Figure 7.37 shows the heat capacity of a Bose gas as a function of temperature. In this problem you will calculate the shape of this unusual graph.

(a) Write down an expression for the total energy of a gas of N bosons confined to a volume V, in terms of an integral (analogous to equation 7.122).

(b) For $T < T_c$ you can set $\mu = 0$. Evaluate the integral numerically in this case, then differentiate the result with respect to T to obtain the heat capacity. Compare to Figure 7.37.

(c) Explain why the heat capacity must approach $\frac{3}{2}Nk$ in the high-T limit.

(d) For $T > T_c$ you can evaluate the integral using the values of μ calculated in Problem 7.69. Do this to obtain the energy as a function of temperature, then numerically differentiate the result to obtain the heat capacity. Plot the heat capacity, and check that your graph agrees with Figure 7.37.

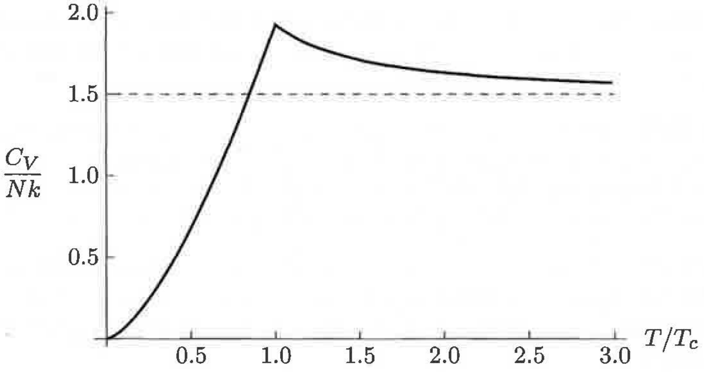

Figure 7.37. Heat capacity of an ideal Bose gas in a three-dimensional box.

Problem 7.71. Starting from the formula for C_V derived in Problem 7.70(b), calculate the entropy, Helmholtz free energy, and pressure of a Bose gas for $T < T_c$. Notice that the pressure is independent of volume; how can this be the case?

Problem 7.72. For a gas of particles confined inside a *two*-dimensional box, the density of states is constant, independent of ϵ (see Problem 7.28). Investigate the behavior of a gas of noninteracting bosons in a two-dimensional box. You should find that the chemical potential remains significantly less than zero as long as T is significantly greater than zero, and hence that there is no abrupt condensation of particles into the ground state. Explain how you know that this is the case, and describe what *does* happen to this system as the temperature decreases. What property must $g(\epsilon)$ have in order for there to be an abrupt Bose-Einstein condensation?

Problem 7.73. Consider a gas of N identical spin-0 bosons confined by an isotropic three-dimensional harmonic oscillator potential. (In the rubidium experiment discussed above, the confining potential was actually harmonic, though not isotropic.) The energy levels in this potential are $\epsilon = nhf$, where n is any nonnegative integer and f is the classical oscillation frequency. The degeneracy of level n is $(n+1)(n+2)/2$.

(a) Find a formula for the density of states, $g(\epsilon)$, for an atom confined by this potential. (You may assume $n \gg 1$.)

(b) Find a formula for the condensation temperature of this system, in terms of the oscillation frequency f.

(c) This potential effectively confines particles inside a volume of roughly the cube of the oscillation amplitude. The oscillation amplitude, in turn, can be estimated by setting the particle's total energy (of order kT) equal to the potential energy of the "spring." Making these associations, and neglecting all factors of 2 and π and so on, show that your answer to part (b) is roughly equivalent to the formula derived in the text for the condensation temperature of bosons confined inside a box with rigid walls.

Problem 7.74. Consider a Bose gas confined in an isotropic harmonic trap, as in the previous problem. For this system, because the energy level structure is much simpler than that of a three-dimensional box, it is feasible to carry out the sum in equation 7.121 *numerically*, without approximating it as an integral.*

(a) Write equation 7.121 for this system as a sum over energy levels, taking degeneracy into account. Replace T and μ with the dimensionless variables $t = kT/hf$ and $c = \mu/hf$.

(b) Program a computer to calculate this sum for any given values of t and c. Show that, for $N = 2000$, equation 7.121 is satisfied at $t = 15$ provided that $c = -10.534$. (Hint: You'll need to include approximately the first 200 energy levels in the sum.)

(c) For the same parameters as in part (b), plot the number of particles in each energy level as a function of energy.

(d) Now reduce t to 14, and adjust the value of c until the sum again equals 2000. Plot the number of particles as a function of energy.

(e) Repeat part (d) for $t = 13$, 12, 11, and 10. You should find that the required value of c increases toward zero but never quite reaches it. Discuss the results in some detail.

---

*This problem is based on an article by Martin Ligare, *American Journal of Physics* **66**, 185–190 (1998).

Problem 7.75. Consider a gas of noninteracting spin-0 bosons at *high* temperatures, when $T \gg T_c$. (Note that "high" in this sense can still mean below 1 K.)

(a) Show that, in this limit, the Bose-Einstein distribution function can be written approximately as

$$\overline{n}_{\text{BE}} = e^{-(\epsilon-\mu)/kT}\left[1 + e^{-(\epsilon-\mu)/kT} + \cdots\right].$$

(b) Keeping only the terms shown above, plug this result into equation 7.122 to derive the first quantum correction to the chemical potential for a gas of bosons.

(c) Use the properties of the grand free energy (Problems 5.23 and 7.7) to show that the pressure of any system is given by $P = (kT/V)\ln\mathcal{Z}$, where $\mathcal{Z}$ is the grand partition function. Argue that, for a gas of noninteracting particles, $\ln\mathcal{Z}$ can be computed as the sum over all modes (or single-particle states) of $\ln\mathcal{Z}_i$, where $\mathcal{Z}_i$ is the grand partition function for the ith mode.

(d) Continuing with the result of part (c), write the sum over modes as an integral over energy, using the density of states. Evaluate this integral explicitly for a gas of noninteracting bosons in the high-temperature limit, using the result of part (b) for the chemical potential and expanding the logarithm as appropriate. When the smoke clears, you should find

$$P = \frac{NkT}{V}\left(1 - \frac{Nv_Q}{4\sqrt{2}V}\right),$$

again neglecting higher-order terms. Thus, quantum statistics results in a *lowering* of the pressure of a boson gas, as one might expect.

(e) Write the result of part (d) in the form of the virial expansion introduced in Problem 1.17, and read off the second virial coefficient, $B(T)$. Plot the predicted $B(T)$ for a hypothetical gas of noninteracting helium-4 atoms.

(f) Repeat this entire problem for a gas of spin-1/2 fermions. (Very few modifications are necessary.) Discuss the results, and plot the predicted virial coefficient for a hypothetical gas of noninteracting helium-3 atoms.

Ten percent or more of a complete stellar inventory consists of white dwarfs, just sitting there, radiating away the thermal (kinetic) energy of their carbon and oxygen nuclei from underneath very thin skins of hydrogen and helium. They will continue this uneventful course until the universe recontracts, their baryons decay, or they collapse to black holes by barrier penetration. (Likely time scales for these three outcomes are 10^{14}, 10^{33}, *and* $10^{10^{76}}$*—years for the first two and for the third one it doesn't matter.)*

—Virginia Trimble, *SLAC Beam Line*
21, 3 (fall, 1991).