(a) Calculate and plot the chemical potential as a function of temperature, for silicon doped with $10^{17}$ phosphorus atoms per $\text{cm}^3$ (as in Problem 7.5). Continue to assume that the conduction electrons can be treated as an ordinary ideal gas.

(b) Discuss whether it is legitimate to assume for this system that the conduction electrons can be treated as an ordinary ideal gas, as opposed to a Fermi gas. Give some numerical examples.

(c) Estimate the temperature at which the number of valence electrons excited to the conduction band would become comparable to the number of conduction electrons from donor impurities. Which source of conduction electrons is more important at room temperature?

**Problem 7.36.** Most spin-1/2 fermions, including electrons and helium-3 atoms, have nonzero magnetic moments. A gas of such particles is therefore paramagnetic. Consider, for example, a gas of free electrons, confined inside a three-dimensional box. The $z$ component of the magnetic moment of each electron is $\pm\mu_B$. In the presence of a magnetic field $B$ pointing in the $z$ direction, each "up" state acquires an additional energy of $-\mu_B B$, while each "down" state acquires an additional energy of $+\mu_B B$.

(a) Explain why you would expect the magnetization of a degenerate electron gas to be substantially less than that of the electronic paramagnets studied in Chapters 3 and 6, for a given number of particles at a given field strength.

(b) Write down a formula for the density of states of this system in the presence of a magnetic field $B$, and interpret your formula graphically.

(c) The magnetization of this system is $\mu_B(N_\uparrow - N_\downarrow)$, where $N_\uparrow$ and $N_\downarrow$ are the numbers of electrons with up and down magnetic moments, respectively. Find a formula for the magnetization of this system at $T = 0$, in terms of $N$, $\mu_B$, $B$, and the Fermi energy.

(d) Find the first temperature-dependent correction to your answer to part (c), in the limit $T \ll T_F$. You may assume that $\mu_B B \ll kT$; this implies that the presence of the magnetic field has negligible effect on the chemical potential $\mu$. (To avoid confusing $\mu_B$ with $\mu$, I suggest using an abbreviation such as $\delta$ for the quantity $\mu_B B$.)

## 7.4 Blackbody Radiation

As a next application of quantum statistics, I'd like to consider the electromagnetic radiation inside some "box" (like an oven or kiln) at a given temperature. First let me discuss what we would expect of such a system in classical (i.e., non-quantum) physics.

### The Ultraviolet Catastrophe

In classical physics, we treat electromagnetic radiation as a continuous "field" that permeates all space. Inside a box, we can think of this field as a combination of various standing-wave patterns, as shown in Figure 7.18. Each standing-wave pattern behaves as a harmonic oscillator with frequency $f = c/\lambda$. Like a mechanical oscillator, each electromagnetic standing wave has two degrees of freedom,

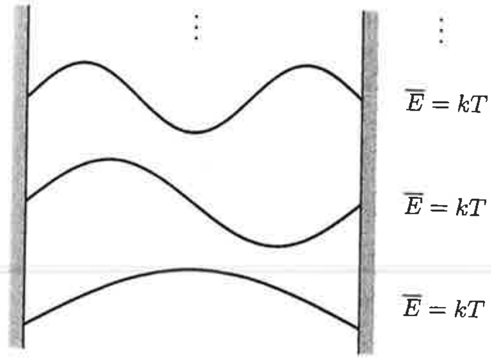


**Figure 7.18.** We can analyze the electromagnetic field in a box as a superposition of standing-wave modes of various wavelengths. Each mode is a harmonic oscillator with some well-defined frequency. Classically, each oscillator should have an average energy of $kT$. Since the total number of modes is infinite, so is the total energy in the box.

with an average thermal energy of $2 \cdot \frac{1}{2}kT$. Since the total number of oscillators in the electromagnetic field is infinite, the total thermal energy should also be infinite. Experimentally, though, you're not blasted with an infinite amount of electromagnetic radiation every time you open the oven door to check the cookies. This disagreement between classical theory and experiment is called the **ultraviolet catastrophe** (because the infinite energy would come mostly from very short wavelengths).

### The Planck Distribution

The solution to the ultraviolet catastrophe comes from quantum mechanics. (Historically, the ultraviolet catastrophe led to the *birth* of quantum mechanics.) In quantum mechanics, a harmonic oscillator can't have just any amount of energy; its allowed energy levels are

$$E_n = 0,\ hf,\ 2hf,\ \ldots. \tag{7.69}$$

(As usual I'm measuring all energies relative to the ground-state energy. See Appendix A for more discussion of this point.) The partition function for a single oscillator is therefore

$$\begin{aligned} Z &= 1 + e^{-\beta hf} + e^{-2\beta hf} + \cdots \\ &= \frac{1}{1 - e^{-\beta hf}}, \end{aligned} \tag{7.70}$$

and the average energy is

$$\overline{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{hf}{e^{hf/kT} - 1}. \tag{7.71}$$

If we think of the energy as coming in "units" of $hf$, then the average *number* of units of energy in the oscillator is

$$\overline{n}_{\rm Pl} = \frac{1}{e^{hf/kT} - 1}. \tag{7.72}$$

This formula is called the **Planck distribution** (after Max Planck).

According to the Planck distribution, short-wavelength modes of the electromagnetic field, with $hf \gg kT$, are *exponentially* suppressed: They are "frozen out," and might as well not exist. Thus the total number of electromagnetic oscillators that effectively contribute to the energy inside the box is finite, and the ultraviolet catastrophe does not occur. Notice that this solution *requires* that the oscillator energies be quantized: It is the size of the energy units, compared to $kT$, that provides the exponential suppression factor.

### Photons

"Units" of energy in the electromagnetic field can also be thought of as *particles*, called **photons**. They are bosons, so the number of them in any "mode" or wave pattern of the field ought to be given by the Bose-Einstein distribution:

$$\overline{n}_{\text{BE}} = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}. \tag{7.73}$$

Here $\epsilon$ is the energy of each particle in the mode, that is, $\epsilon = hf$. Comparison with equation 7.72 therefore requires

$$\mu = 0 \qquad \text{for photons.} \tag{7.74}$$

But why should this be true? I'll give you two reasons, both based on the fact that photons can be created or destroyed in any quantity; their total number is not conserved.

First consider the Helmholtz free energy, which must attain the minimum possible value at equilibrium with $T$ and $V$ held fixed. In a system of photons, the number $N$ of particles is not constrained, but rather takes whatever value will minimize $F$. If $N$ then changes infinitesimally, $F$ should be unchanged:

$$\left(\frac{\partial F}{\partial N}\right)_{T,V} = 0 \qquad \text{(at equilibrium).} \tag{7.75}$$

But this partial derivative is precisely equal to the chemical potential.

A second argument makes use of the condition for chemical equilibrium derived in Section 5.6. Consider a typical reaction in which a photon ($\gamma$) is created or absorbed by an electron:

$$e \longleftrightarrow e + \gamma. \tag{7.76}$$

As we saw in Section 5.6, the equilibrium condition for such a reaction is the same as the reaction equation, with the name of each species replaced by its chemical potential. In this case,

$$\mu_e = \mu_e + \mu_\gamma \qquad \text{(at equilibrium).} \tag{7.77}$$

In other words, the chemical potential for photons is zero.

By either argument, the chemical potential for a "gas" of photons inside a box at fixed temperature is zero, so the Bose-Einstein distribution reduces to the Planck distribution, as required.

## Summing over Modes

The Planck distribution tells us how many photons are in any single "mode" (or "single-particle state") of the electromagnetic field. Next we might want to know the *total* number of photons inside the box, and also the total *energy* of all the photons. To compute either one, we have to sum over all possible states, just as we did for electrons. I'll compute the total energy, and let you compute the total number of photons in Problem 7.44.

Let's start in one dimension, with a "box" of length $L$. The allowed wavelengths and momenta are the same for photons as for any other particles:

$$\lambda = \frac{2L}{n}; \qquad p = \frac{hn}{2L}. \tag{7.78}$$

(Here $n$ is a positive integer that labels which mode we're talking about, not to be confused with $\overline{n}_{\text{Pl}}$, the average number of photons in a given mode.) Photons, however, are ultrarelativistic particles, so their energies are given by

$$\epsilon = pc = \frac{hcn}{2L} \tag{7.79}$$

instead of $\epsilon = p^2/2m$. (You can also derive this result straight from the Einstein relation $\epsilon = hf$ between a photon's energy and its frequency. For light, $f = c/\lambda$, so $\epsilon = hc/\lambda = hcn/2L$.)

In three dimensions, momentum becomes a vector, with each component given by $h/2L$ times some integer. The energy is $c$ times the *magnitude* of the momentum vector:

$$\epsilon = c\sqrt{p_x^2 + p_y^2 + p_z^2} = \frac{hc}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{hcn}{2L}, \tag{7.80}$$

where in the last expression I'm using $n$ for the magnitude of the $\vec{n}$ vector, as in Section 7.3.

Now the average energy in any particular mode is equal to $\epsilon$ times the occupancy of that mode, and the occupancy is given by the Planck distribution. To get the total energy in all modes, we sum over $n_x$, $n_y$, and $n_z$. We also need to slip in a factor of 2, since each wave shape can hold photons with two independent polarizations. So the total energy is

$$U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z} \epsilon\,\overline{n}_{\text{Pl}}(\epsilon) = \sum_{n_x,n_y,n_z} \frac{hcn}{L}\frac{1}{e^{hcn/2LkT} - 1}. \tag{7.81}$$

As in Section 7.3, we can convert the sums to integrals and carry out the integration in spherical coordinates (see Figure 7.11). This time, however, the upper limit on the integration over $n$ is infinity:

$$U = \int_0^\infty dn \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi\, n^2 \sin\theta\, \frac{hcn}{L}\frac{1}{e^{hcn/2LkT} - 1}. \tag{7.82}$$

Again the angular integrals give $\pi/2$, the surface area of an eighth of a unit sphere.

### The Planck Spectrum

The integral over $n$ looks a little nicer if we change variables to the photon energy, $\epsilon = hcn/2L$. We then get an overall factor of $L^3 = V$, so the total energy per unit volume is

$$\frac{U}{V} = \int_0^\infty \frac{8\pi\epsilon^3/(hc)^3}{e^{\epsilon/kT} - 1}\, d\epsilon. \tag{7.83}$$

Here the integrand has a nice interpretation: It is the energy density per unit photon energy, or the **spectrum** of the photons:

$$u(\epsilon) = \frac{8\pi}{(hc)^3}\frac{\epsilon^3}{e^{\epsilon/kT} - 1}. \tag{7.84}$$

This function, first derived by Planck, gives the relative intensity of the radiation as a function of photon energy (or as a function of frequency, if you change variables again to $f = \epsilon/h$). If you integrate $u(\epsilon)$ from $\epsilon_1$ to $\epsilon_2$, you get the energy per unit volume within that range of photon energies.

To actually evaluate the integral over $\epsilon$, it's convenient to change variables again, to $x = \epsilon/kT$. Then equation 7.83 becomes

$$\frac{U}{V} = \frac{8\pi(kT)^4}{(hc)^3}\int_0^\infty \frac{x^3}{e^x - 1}\, dx. \tag{7.85}$$

The integrand is still proportional to the Planck spectrum; this function is plotted in Figure 7.19. The spectrum peaks at $x = 2.82$, or $\epsilon = 2.82kT$. Not surprisingly, higher temperatures tend to give higher photon energies. (This fact is called

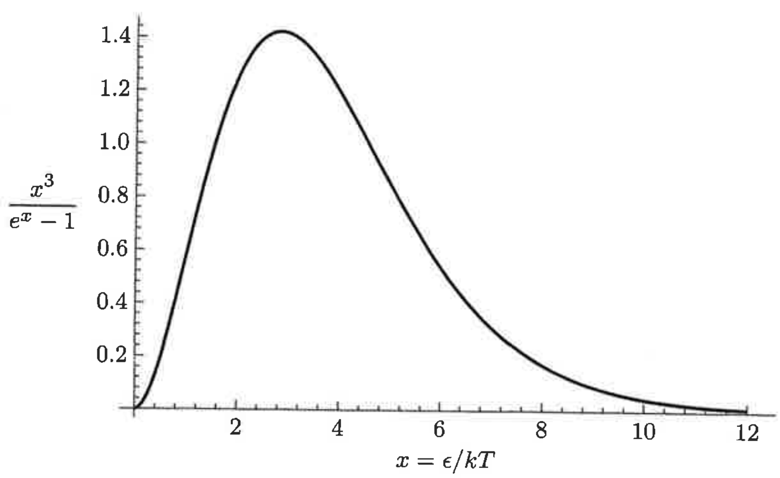


**Figure 7.19.** The Planck spectrum, plotted in terms of the dimensionless variable $x = \epsilon/kT = hf/kT$. The area under any portion of this graph, multiplied by $8\pi(kT)^4/(hc)^3$, equals the energy density of electromagnetic radiation within the corresponding frequency (or photon energy) range; see equation 7.85.

**Wien's law.**) You can measure the temperature inside an oven (or more likely, a kiln) by letting a bit of the radiation out and looking at its color. For instance, a typical clay-firing temperature of 1500 K gives a spectrum that peaks at $\epsilon = 0.36$ eV, in the near infrared. (Visible-light photons have higher energies, in the range of about 2–3 eV.)

**Problem 7.37.** Prove that the peak of the Planck spectrum is at $x = 2.82$.

**Problem 7.38.** It's not obvious from Figure 7.19 how the Planck spectrum changes as a function of temperature. To examine the temperature dependence, make a quantitative plot of the function $u(\epsilon)$ for $T = 3000$ K and $T = 6000$ K (both on the same graph). Label the horizontal axis in electron-volts.

**Problem 7.39.** Change variables in equation 7.83 to $\lambda = hc/\epsilon$, and thus derive a formula for the photon spectrum as a function of wavelength. Plot this spectrum, and find a numerical formula for the wavelength where the spectrum peaks, in terms of $hc/kT$. Explain why the peak does not occur at $hc/(2.82kT)$.

**Problem 7.40.** Starting from equation 7.83, derive a formula for the density of states of a photon gas (or any other gas of ultrarelativistic particles having two polarization states). Sketch this function.

**Problem 7.41.** Consider any two internal states, $s_1$ and $s_2$, of an atom. Let $s_2$ be the higher-energy state, so that $E(s_2) - E(s_1) = \epsilon$ for some positive constant $\epsilon$. If the atom is currently in state $s_2$, then there is a certain probability per unit time for it to spontaneously decay down to state $s_1$, emitting a photon with energy $\epsilon$. This probability per unit time is called the **Einstein $A$ coefficient**:

$$A = \text{probability of spontaneous decay per unit time.}$$

On the other hand, if the atom is currently in state $s_1$ and we shine light on it with frequency $f = \epsilon/h$, then there is a chance that it will absorb a photon, jumping into state $s_2$. The probability for this to occur is proportional not only to the amount of time elapsed but also to the intensity of the light, or more precisely, the energy density of the light per unit frequency, $u(f)$. (This is the function which, when integrated over any frequency interval, gives the energy per unit volume within that frequency interval. For our atomic transition, all that matters is the value of $u(f)$ at $f = \epsilon/h$.) The probability of absorbing a photon, per unit time per unit intensity, is called the **Einstein $B$ coefficient**:

$$B = \frac{\text{probability of absorption per unit time}}{u(f)}.$$

Finally, it is also possible for the atom to make a *stimulated* transition from $s_2$ down to $s_1$, again with a probability that is proportional to the intensity of light at frequency $f$. (Stimulated emission is the fundamental mechanism of the laser: Light Amplification by Stimulated Emission of Radiation.) Thus we define a third coefficient, $B'$, that is analogous to $B$:

$$B' = \frac{\text{probability of stimulated emission per unit time}}{u(f)}.$$

As Einstein showed in 1917, knowing any one of these three coefficients is as good as knowing them all.

(a) Imagine a collection of many of these atoms, such that $N_1$ of them are in state $s_1$ and $N_2$ are in state $s_2$. Write down a formula for $dN_1/dt$ in terms of $A$, $B$, $B'$, $N_1$, $N_2$, and $u(f)$.

(b) Einstein's trick is to imagine that these atoms are bathed in *thermal* radiation, so that $u(f)$ is the Planck spectral function. At equilibrium, $N_1$ and $N_2$ should be constant in time, with their ratio given by a simple Boltzmann factor. Show, then, that the coefficients must be related by

$$B' = B \quad \text{and} \quad \frac{A}{B} = \frac{8\pi h f^3}{c^3}.$$

### Total Energy

Enough about the spectrum—what about the total electromagnetic energy inside the box? Equation 7.85 is essentially the final answer, except for the integral over $x$, which is just some dimensionless number. From Figure 7.19 you can estimate that this number is about 6.5; a beautiful but very tricky calculation (see Appendix B) gives it exactly as $\pi^4/15$. Therefore the total energy density, summing over all frequencies, is

$$\frac{U}{V} = \frac{8\pi^5 (kT)^4}{15(hc)^3}. \tag{7.86}$$

The most important feature of this result is its dependence on the *fourth* power of the temperature. If you double the temperature of your oven, the amount of electromagnetic energy inside increases by a factor of $2^4 = 16$.

Numerically, the total electromagnetic energy inside a typical oven is quite small. At cookie-baking temperature, 375°F or about 460 K, the energy per unit volume comes out to $3.5 \times 10^{-5}$ J/m$^3$. This is *tiny* compared to the thermal energy of the air inside the oven.

Formula 7.86 may look complicated, but you could have guessed the answer, aside from the numerical coefficient, by dimensional analysis. The average energy per photon must be something of order $kT$, so the total energy must be proportional to $NkT$, where $N$ is the total number of photons. Since $N$ is extensive, it must be proportional to the volume $V$ of the container; thus the total energy must be of the form

$$U = (\text{constant}) \cdot \frac{V\,kT}{\ell^3}, \tag{7.87}$$

where $\ell$ is something with units of length. (If you want, you can pretend that each photon occupies a volume of $\ell^3$.) But the only relevant length in the problem is the typical de Broglie wavelength of the photons, $\lambda = h/p = hc/E \propto hc/kT$. Plugging this in for $\ell$ yields equation 7.86, aside from the factor of $8\pi^5/15$.

**Problem 7.42.** Consider the electromagnetic radiation inside a kiln, with a volume of 1 m$^3$ and a temperature of 1500 K.

(a) What is the total energy of this radiation?

(b) Sketch the spectrum of the radiation as a function of photon energy.

(c) What fraction of all the energy is in the *visible* portion of the spectrum, with wavelengths between 400 nm and 700 nm?

**Problem 7.43.** At the surface of the sun, the temperature is approximately 5800 K.

(a) How much energy is contained in the electromagnetic radiation filling a cubic meter of space at the sun's surface?

(b) Sketch the spectrum of this radiation as a function of photon energy. Mark the region of the spectrum that corresponds to visible wavelengths, between 400 nm and 700 nm.

(c) What fraction of the energy is in the visible portion of the spectrum? (Hint: Do the integral numerically.)

## Entropy of a Photon Gas

Besides the total energy of a photon gas, we might want to know a number of other quantities, for instance, the total number of photons present or the total entropy. These two quantities turn out to be equal, up to a constant factor. Let me now compute the entropy.

The easiest way to compute the entropy is from the heat capacity. For a box of thermal photons with volume $V$,

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 4aT^3, \tag{7.88}$$

where $a$ is an abbreviation for $8\pi^5k^4V/15(hc)^3$. This expression is good all the way down to absolute zero, so we can integrate it to find the absolute entropy. Introducing the symbol $T'$ for the integration variable,

$$S(T) = \int_0^T \frac{C_V(T')}{T'}\,dT' = 4a\int_0^T (T')^2\,dT' = \frac{4}{3}aT^3 = \frac{32\pi^5}{45}V\left(\frac{kT}{hc}\right)^3 k. \tag{7.89}$$

The total *number* of photons is given by the same formula, with a different numerical coefficient, and without the final $k$ (see Problem 7.44).

## The Cosmic Background Radiation

The grandest example of a photon gas is the radiation that fills the entire observable universe, with an almost perfect thermal spectrum at a temperature of 2.73 K. Interpreting this temperature is a bit tricky, however: There is no longer any mechanism to keep the photons in thermal equilibrium with each other or with anything else; the radiation is instead thought to be left over from a time when the universe was filled with ionized gas that interacted strongly with electromagnetic radiation. At that time, the temperature was more like 3000 K; since then the universe has expanded a thousandfold in all directions, and the photon wavelengths have been stretched out accordingly (Doppler-shifted, if you care to think of it this way), preserving the shape of the spectrum but shifting the effective temperature down to 2.73 K.

The photons making up the cosmic background radiation have rather low energies: The spectrum peaks at $\epsilon = 2.82kT = 6.6 \times 10^{-4}$ eV. This corresponds to wavelengths of about a millimeter, in the far infrared. These wavelengths don't penetrate our atmosphere, but the long-wavelength tail of the spectrum, in the microwave region of a few centimeters, can be detected without much difficulty. It was discovered accidentally by radio astronomers in 1965. Figure 7.20 shows a more recent set of measurements over a wide range of wavelengths, made from above earth's atmosphere by the *Cosmic Background Explorer* satellite.

According to formula 7.86, the total energy in the cosmic background radiation is only 0.26 $\text{MeV/m}^3$. This is to be contrasted with the average energy density of ordinary matter, which on cosmic scales is of the order of a proton per cubic meter or 1000 $\text{MeV/m}^3$. (Ironically, the density of the exotic background radiation is known to three significant figures, while the average density of ordinary matter is uncertain by nearly a factor of 10.) On the other hand, the *entropy* of the background radiation is much greater than that of ordinary matter: According to equation 7.89, every cubic meter of space contains a photon entropy of $(2.89\times10^9)k$, nearly three billion "units" of entropy. The entropy of ordinary matter is not easy to calculate precisely, but if we pretend that this matter is an ordinary ideal gas we can estimate that its entropy is $Nk$ times some small number, in other words, only a few $k$ per cubic meter.

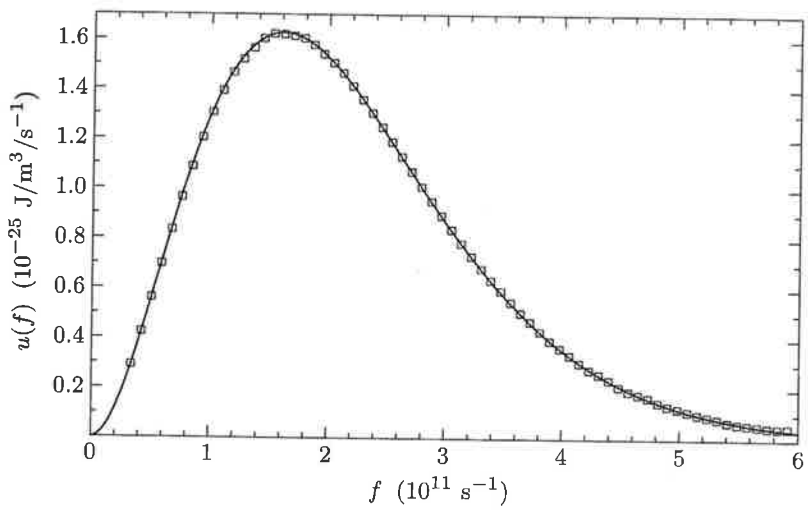


**Figure 7.20.** Spectrum of the cosmic background radiation, as measured by the *Cosmic Background Explorer* satellite. Plotted vertically is the energy density per unit frequency, in SI units. Note that a frequency of $3\times10^{11}\ \text{s}^{-1}$ corresponds to a wavelength of $\lambda = c/f = 1.0$ mm. Each square represents a measured data point. The point-by-point uncertainties are too small to show up on this scale; the size of the squares instead represents a liberal estimate of the uncertainty due to systematic effects. The solid curve is the theoretical Planck spectrum, with the temperature adjusted to 2.735 K to give the best fit. From J. C. Mather et al., *Astrophysical Journal Letters* **354**, L37 (1990); adapted courtesy of NASA/GSFC and the COBE Science Working Group. Subsequent measurements from this experiment and others now give a best-fit temperature of $2.728 \pm 0.002$ K.

**Problem 7.44.** Number of photons in a photon gas.

(a) Show that the number of photons in equilibrium in a box of volume $V$ at temperature $T$ is

$$N = 8\pi V\left(\frac{kT}{hc}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1}\,dx.$$

The integral cannot be done analytically; either look it up in a table or evaluate it numerically.

(b) How does this result compare to the formula derived in the text for the entropy of a photon gas? (What is the entropy per photon, in terms of $k$?)

(c) Calculate the number of photons per cubic meter at the following temperatures: 300 K; 1500 K (a typical kiln); 2.73 K (the cosmic background radiation).

**Problem 7.45.** Use the formula $P = -(\partial U/\partial V)_{S,N}$ to show that the pressure of a photon gas is 1/3 times the energy density ($U/V$). Compute the pressure exerted by the radiation inside a kiln at 1500 K, and compare to the ordinary gas pressure exerted by the air. Then compute the pressure of the radiation at the center of the sun, where the temperature is 15 million K. Compare to the gas pressure of the ionized hydrogen, whose density is approximately $10^5$ kg/m$^3$.

**Problem 7.46.** Sometimes it is useful to know the free energy of a photon gas.

(a) Calculate the (Helmholtz) free energy directly from the definition $F = U - TS$. (Express the answer in terms of $T$ and $V$.)

(b) Check the formula $S = -(\partial F/\partial T)_V$ for this system.

(c) Differentiate $F$ with respect to $V$ to obtain the pressure of a photon gas. Check that your result agrees with that of the previous problem.

(d) A more interesting way to calculate $F$ is to apply the formula $F = -kT \ln Z$ separately to each mode (that is, each effective oscillator), then sum over all modes. Carry out this calculation, to obtain

$$F = 8\pi V \frac{(kT)^4}{(hc)^3} \int_0^\infty x^2 \ln(1 - e^{-x})\,dx.$$

Integrate by parts, and check that your answer agrees with part (a).

**Problem 7.47.** In the text I claimed that the universe was filled with ionized gas until its temperature cooled to about 3000 K. To see why, assume that the universe contains only photons and hydrogen atoms, with a constant ratio of $10^9$ photons per hydrogen atom. Calculate and plot the fraction of atoms that were ionized as a function of temperature, for temperatures between 0 and 6000 K. How does the result change if the ratio of photons to atoms is $10^8$, or $10^{10}$? (Hint: Write everything in terms of dimensionless variables such as $t = kT/I$, where $I$ is the ionization energy of hydrogen.)

**Problem 7.48.** In addition to the cosmic background radiation of photons, the universe is thought to be permeated with a background radiation of neutrinos ($\nu$) and antineutrinos ($\bar{\nu}$), currently at an effective temperature of 1.95 K. There are three species of neutrinos, each of which has an antiparticle, with only one allowed polarization state for each particle or antiparticle. For parts (a) through (c) below, assume that all three species are exactly massless.

**(a)** It is reasonable to assume that for each species, the concentration of neutrinos equals the concentration of antineutrinos, so that their chemical potentials are equal: $\mu_\nu = \mu_{\bar{\nu}}$. Furthermore, neutrinos and antineutrinos can be produced and annihilated in pairs by the reaction

$$\nu + \bar{\nu} \leftrightarrow 2\gamma$$

(where $\gamma$ is a photon). Assuming that this reaction is at equilibrium (as it would have been in the very early universe), prove that $\mu = 0$ for both the neutrinos and the antineutrinos.

**(b)** If neutrinos are massless, they must be highly relativistic. They are also fermions: They obey the exclusion principle. Use these facts to derive a formula for the total energy density (energy per unit volume) of the neutrino-antineutrino background radiation. (Hint: There are very few differences between this "neutrino gas" and a photon gas. Antiparticles still have positive energy, so to include the antineutrinos all you need is a factor of 2. To account for the three species, just multiply by 3.) To evaluate the final integral, first change to a dimensionless variable and then use a computer or look it up in a table or consult Appendix B.

**(c)** Derive a formula for the *number* of neutrinos per unit volume in the neutrino background radiation. Evaluate your result numerically for the present neutrino temperature of 1.95 K.

**(d)** It is possible that neutrinos have very small, but nonzero, masses. This wouldn't have affected the production of neutrinos in the early universe, when $mc^2$ would have been negligible compared to typical thermal energies. But today, the total mass of all the background neutrinos could be significant. Suppose, then, that just one of the three species of neutrinos (and the corresponding antineutrino) has a nonzero mass $m$. What would $mc^2$ have to be (in eV), in order for the total mass of neutrinos in the universe to be comparable to the total mass of ordinary matter?

**Problem 7.49.** For a brief time in the early universe, the temperature was hot enough to produce large numbers of electron-positron pairs. These pairs then constituted a third type of "background radiation," in addition to the photons and neutrinos (see Figure 7.21). Like neutrinos, electrons and positrons are fermions. Unlike neutrinos, electrons and positrons are known to be massive (each with the same mass), and each has two independent polarization states. During the time period of interest the densities of electrons and positrons were approximately equal, so it is a good approximation to set the chemical potentials equal to zero as in

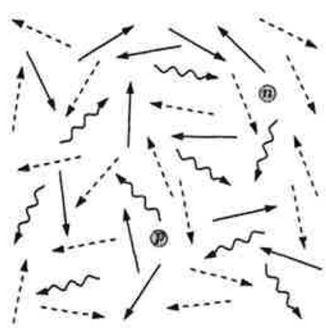


**Figure 7.21.** When the temperature was greater than the electron mass times $c^2/k$, the universe was filled with three types of radiation: electrons and positrons (solid arrows); neutrinos (dashed); and photons (wavy). Bathed in this radiation were a few protons and neutrons, roughly one for every billion radiation particles.

the previous problem. Recall from special relativity that the energy of a massive particle is $\epsilon = \sqrt{(pc)^2 + (mc^2)^2}$.

**(a)** Show that the energy density of electrons and positrons at temperature $T$ is given by

$$\frac{U}{V} = \frac{16\pi(kT)^4}{(hc)^3}\,u(T),$$

where

$$u(T) = \int_0^\infty \frac{x^2\sqrt{x^2 + (mc^2/kT)^2}}{e^{\sqrt{x^2+(mc^2/kT)^2}} + 1}\,dx.$$

**(b)** Show that $u(T)$ goes to zero when $kT \ll mc^2$, and explain why this is a reasonable result.

**(c)** Evaluate $u(T)$ in the limit $kT \gg mc^2$, and compare to the result of the previous problem for the neutrino radiation.

**(d)** Use a computer to calculate and plot $u(T)$ at intermediate temperatures.

**(e)** Use the method of Problem 7.46, part (d), to show that the free energy density of the electron-positron radiation is

$$\frac{F}{V} = -\frac{16\pi(kT)^4}{(hc)^3}\,f(T),$$

where

$$f(T) = \int_0^\infty x^2 \ln\left(1 + e^{-\sqrt{x^2+(mc^2/kT)^2}}\right) dx.$$

Evaluate $f(T)$ in both limits, and use a computer to calculate and plot $f(T)$ at intermediate temperatures.

**(f)** Write the entropy of the electron-positron radiation in terms of the functions $u(T)$ and $f(T)$. Evaluate the entropy explicitly in the high-$T$ limit.

**Problem 7.50.** The results of the previous problem can be used to explain why the current temperature of the cosmic neutrino background (Problem 7.48) is 1.95 K rather than 2.73 K. Originally the temperatures of the photons and the neutrinos would have been equal, but as the universe expanded and cooled, the interactions of neutrinos with other particles soon became negligibly weak. Shortly thereafter, the temperature dropped to the point where $kT/c^2$ was no longer much greater than the electron mass. As the electrons and positrons disappeared during the next few minutes, they "heated" the photon radiation but not the neutrino radiation.

**(a)** Imagine that the universe has some finite total volume $V$, but that $V$ is increasing with time. Write down a formula for the total entropy of the electrons, positrons, and photons as a function of $V$ and $T$, using the auxilliary functions $u(T)$ and $f(T)$ introduced in the previous problem. Argue that this total entropy would have been conserved in the early universe, assuming that no other species of particles interacted with these.

**(b)** The entropy of the neutrino radiation would have been separately conserved during this time period, because the neutrinos were unable to interact with anything. Use this fact to show that the neutrino temperature $T_\nu$ and the photon temperature $T$ are related by

$$\left(\frac{T}{T_\nu}\right)^3 \left[\frac{2\pi^4}{45} + u(T) + f(T)\right] = \text{constant}$$

as the universe expands and cools. Evaluate the constant by assuming that $T = T_\nu$ when the temperatures are very high.

(c) Calculate the ratio $T/T_\nu$ in the limit of low temperature, to confirm that the present neutrino temperature should be 1.95 K.

(d) Use a computer to plot the ratio $T/T_\nu$ as a function of $T$, for $kT/mc^2$ ranging from 0 to 3.*

### Photons Escaping through a Hole

So far in this section I have analyzed the gas of photons *inside* an oven or any other box in thermal equilibrium. Eventually, though, we'd like to understand the photons *emitted* by a hot object. To begin, let's ask what happens if you start with a photon gas in a box, then poke a hole in the box to let some photons out (see Figure 7.22).

All photons travel at the same speed (in vacuum), regardless of their wavelengths. So low-energy photons will escape through the hole with the same probability as high-energy photons, and thus the spectrum of the photons coming out will look the same as the spectrum of the photons inside. What's harder to figure out is the total *amount* of radiation that escapes; the calculation doesn't involve much physics, but the geometry is rather tricky.

The photons that escape now, during a time interval $dt$, were once pointed at the hole from somewhere within a hemispherical shell, as shown in Figure 7.23. The radius $R$ of the shell depends on how long ago we're looking, while the thickness of the shell is $c\,dt$. I'll use spherical coordinates to label various points on the shell, as shown. The angle $\theta$ ranges from 0, at the left end of the shell, to $\pi/2$, at the extreme edges on the right. There's also an azimuthal angle $\phi$, not shown, which ranges from 0 to $2\pi$ as you go from the top edge of the shell into the page, down

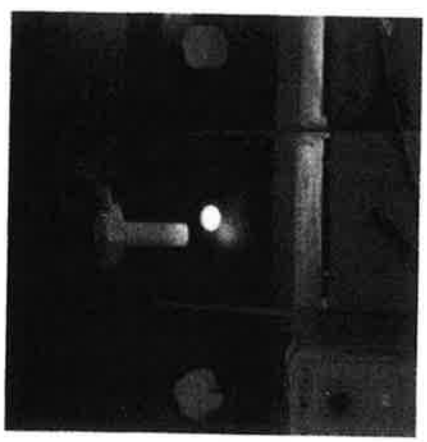

**Figure 7.22.** When you open a hole in a container filled with radiation (here a kiln), the spectrum of the light that escapes is the same as the spectrum of the light inside. The total amount of energy that escapes is proportional to the size of the hole and to the amount of time that passes.

*Now that you've finished this problem, you'll find it relatively easy to work out the dynamics of the early universe, to determine *when* all this happened. The basic idea is to assume that the universe is expanding at "escape velocity." Everything you need to know is in Weinberg (1977).

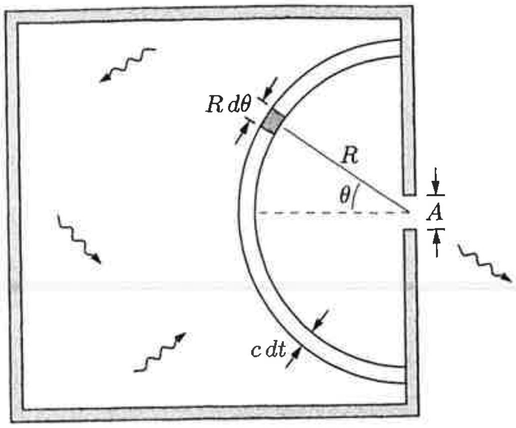


**Figure 7.23.** The photons that escape now were once somewhere within a hemispherical shell inside the box. From a given point in this shell, the probability of escape depends on the distance from the hole and the angle $\theta$.

to the bottom, out of the page, and back to the top.

Now consider the shaded chunk of the shell shown Figure 7.23. Its volume is

$$\text{volume of chunk} = (R\,d\theta) \times (R\sin\theta\,d\phi) \times (c\,dt). \tag{7.90}$$

(The depth of the chunk, perpendicular to the page, is $R\sin\theta\,d\phi$, since $R\sin\theta$ is the radius of a ring of constant $\theta$ swept out as $\phi$ ranges from 0 to $2\pi$.) The energy density of the photons within this chunk is given by equation 7.86:

$$\frac{U}{V} = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}. \tag{7.91}$$

In what follows I'll simply call this quantity $U/V$; the total energy in the chunk is thus

$$\text{energy in chunk} = \frac{U}{V}\,c\,dt\,R^2\sin\theta\,d\theta\,d\phi. \tag{7.92}$$

But not all the energy in this chunk of space will escape through the hole, because most of the photons are pointed in the wrong direction. The probability of a photon being pointed in the *right* direction is equal to the apparent area of the hole, as viewed from the chunk, divided by the total area of an imaginary sphere of radius $R$ centered on the chunk:

$$\text{probability of escape} = \frac{A\cos\theta}{4\pi R^2}. \tag{7.93}$$

Here $A$ is the area of the hole, and $A\cos\theta$ is its foreshortened area, as seen from the chunk. The amount of energy that escapes from this chunk is therefore

$$\text{energy escaping from chunk} = \frac{A\cos\theta}{4\pi}\frac{U}{V}\,c\,dt\,\sin\theta\,d\theta\,d\phi. \tag{7.94}$$

To find the *total* energy that escapes through the hole in the time interval $dt$, we just integrate over $\theta$ and $\phi$:

$$\begin{aligned}\text{total energy escaping} &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, \frac{A\cos\theta}{4\pi}\, \frac{U}{V}\, c\,dt\, \sin\theta \\ &= 2\pi\, \frac{A}{4\pi}\, \frac{U}{V}\, c\,dt \int_0^{\pi/2} \cos\theta\, \sin\theta\, d\theta \qquad (7.95) \\ &= \frac{A}{4}\, \frac{U}{V}\, c\,dt.\end{aligned}$$

The amount of energy that escapes is naturally proportional to the area $A$ of the hole, and also to the duration $dt$ of the time interval. If we divide by these quantities we get the *power* emitted per unit area:

$$\text{power per unit area} = \frac{c}{4}\frac{U}{V}. \qquad (7.96)$$

Aside from the factor of 1/4, you could have guessed this result using dimensional analysis: To turn energy/volume into power/area, you have to multiply by something with units of distance/time, and the only relevant speed in the problem is the speed of light.

Plugging in formula 7.91 for the energy density inside the box, we obtain the more explicit result

$$\text{power per unit area} = \frac{2\pi^5}{15}\frac{(kT)^4}{h^3c^2} = \sigma T^4, \qquad (7.97)$$

where $\sigma$ is known as the **Stefan-Boltzmann constant,**

$$\sigma = \frac{2\pi^5 k^4}{15h^3c^2} = 5.67\times 10^{-8}\ \frac{\mathrm{W}}{\mathrm{m^2\,K^4}}. \qquad (7.98)$$

(This number isn't hard to memorize: Just think "5–6–7–8," and don't forget the minus sign.) The dependence of the power radiated on the fourth power of the temperature is known as **Stefan's law**, and was discovered empirically in 1879.

## Radiation from Other Objects

Although I derived Stefan's law for photons emitted from a hole in a box, it also applies to photons emitted by any nonreflecting ("black") surface at temperature $T$. Such radiation is therefore called **blackbody radiation**. The proof that a black object emits photons exactly as does a hole in a box is amazingly simple.

Suppose you have a hole in a box, on one hand, and a black object, on the other hand, both at the same temperature, facing each other as in Figure 7.24. Each object emits photons, some of which are absorbed by the other. If the objects are the same size, each will absorb the same fraction of the other's radiation. Now suppose that the blackbody does *not* emit the same amount of power as the hole; perhaps

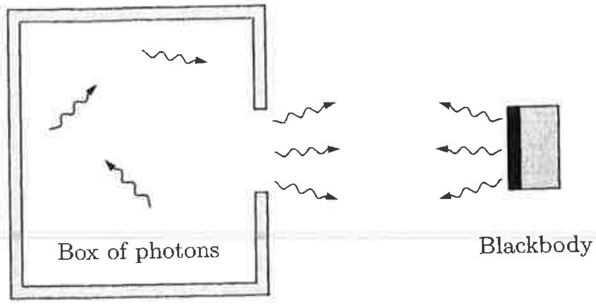


**Figure 7.24.** A thought experiment to demonstrate that a perfectly black surface emits radiation identical to that emitted by a hole in a box of thermal photons.

it emits somewhat less. Then more energy will flow from the hole to the blackbody than from the blackbody to the hole, and the blackbody will gradually get hotter. Oops! This process would violate the second law of thermodynamics. And if the blackbody emits *more* radiation than the hole, then the blackbody gradually cools off while the box with the hole gets hotter; again, this can't happen.

So the total power emitted by the blackbody, per unit area at any given temperature, must be the same as that emitted by the hole. But we can say more. Imagine inserting a filter, which allows only a certain range of wavelengths to pass through, between the hole and the blackbody. Again, if one object emits more radiation at these wavelengths than the other, its temperature will decrease while the other's temperature increases, in violation of the second law. Thus the entire spectrum of radiation emitted by the blackbody must be the same as for the hole.

If an object is *not* black, so that it reflects some photons instead of absorbing them, things get a bit more complicated. Let's say that out of every three photons (at some given wavelength) that hit the object, it reflects one back and absorbs the other two. Now, in order to remain in thermal equilibrium with the hole, it only needs to emit two photons, which join the reflected photon on its way back. More generally, if $e$ is the fraction of photons absorbed (at some given wavelength), then $e$ is also the fraction emitted, in comparison to a perfect blackbody. This number $e$ is called the **emissivity** of the material. It equals 1 for a perfect blackbody, and equals 0 for a perfectly reflective surface. Thus, a good reflector is a poor emitter, and vice versa. Generally the emissivity depends upon the wavelength of the light, so the spectrum of radiation emitted will differ from a perfect blackbody spectrum. If we use a weighted average of $e$ over all relevant wavelengths, then the total power radiated by an object can be written

$$\text{power} = \sigma e A T^4, \tag{7.99}$$

where $A$ is the object's surface area.

**Problem 7.51.** The tungsten filament of an incandescent light bulb has a temperature of approximately 3000 K. The emissivity of tungsten is approximately 1/3, and you may assume that it is independent of wavelength.

(a) If the bulb gives off a total of 100 watts, what is the surface area of its filament in square millimeters?

(b) At what value of the photon energy does the peak in the bulb's spectrum occur? What is the wavelength corresponding to this photon energy?

(c) Sketch (or use a computer to plot) the spectrum of light given off by the filament. Indicate the region on the graph that corresponds to visible wavelengths, between 400 and 700 nm.

(d) Calculate the fraction of the bulb's energy that comes out as visible light. (Do the integral numerically on a calculator or computer.) Check your result qualitatively from the graph of part (c).

(e) To increase the efficiency of an incandescent bulb, would you want to raise or lower the temperature? (Some incandescent bulbs *do* attain slightly higher efficiency by using a different temperature.)

(f) Estimate the maximum possible efficiency (i.e., fraction of energy in the visible spectrum) of an incandescent bulb, and the corresponding filament temperature. Neglect the fact that tungsten melts at 3695 K.

**Problem 7.52.**

(a) Estimate (roughly) the total power radiated by your body, neglecting any energy that is returned by your clothes and environment. (Whatever the color of your skin, its emissivity at infrared wavelengths is quite close to 1; almost any nonmetal is a near-perfect blackbody at these wavelengths.)

(b) Compare the total energy radiated by your body in one day (expressed in kilocalories) to the energy in the food you eat. Why is there such a large discrepancy?

(c) The sun has a mass of $2 \times 10^{30}$ kg and radiates energy at a rate of $3.9 \times 10^{26}$ watts. Which puts out more power *per units mass*—the sun or your body?

**Problem 7.53.** A black hole is a blackbody if ever there was one, so it should emit blackbody radiation, called **Hawking radiation**. A black hole of mass $M$ has a total energy of $Mc^2$, a surface area of $16\pi G^2 M^2/c^4$, and a temperature of $\hbar c^3/16\pi^2 kGM$ (as shown in Problem 3.7).

(a) Estimate the typical wavelength of the Hawking radiation emitted by a one-solar-mass ($2 \times 10^{30}$ kg) black hole. Compare your answer to the size of the black hole.

(b) Calculate the total power radiated by a one-solar-mass black hole.

(c) Imagine a black hole in empty space, where it emits radiation but absorbs nothing. As it loses energy, its mass must decrease; one could say it "evaporates." Derive a differential equation for the mass as a function of time, and solve this equation to obtain an expression for the lifetime of a black hole in terms of its initial mass.

(d) Calculate the lifetime of a one-solar-mass black hole, and compare to the estimated age of the known universe ($10^{10}$ years).

(e) Suppose that a black hole that was created early in the history of the universe finishes evaporating today. What was its initial mass? In what part of the electromagnetic spectrum would most of its radiation have been emitted?

### The Sun and the Earth

From the amount of solar radiation received by the earth (1370 W/m$^2$, known as the **solar constant**) and the earth's distance from the sun (150 million kilometers), it's pretty easy to calculate the sun's total energy output or **luminosity**: $3.9 \times 10^{26}$ watts. The sun's radius is a little over 100 times the earth's: $7.0 \times 10^8$ m; so its surface area is $6.1 \times 10^{18}$ m$^2$. From this information, assuming an emissivity of 1 (which is not terribly accurate but good enough for our purposes), we can calculate the sun's surface temperature:

$$T = \left(\frac{\text{luminosity}}{\sigma A}\right)^{1/4} = 5800 \text{ K}. \tag{7.100}$$

Knowing the temperature, we can predict that the spectrum of sunlight should peak at a photon energy of

$$\epsilon = 2.82\, kT = 1.41 \text{ eV}, \tag{7.101}$$

which corresponds to a wavelength of 880 nm, in the near infrared. This is a testable prediction, and it agrees with experiment: The sun's spectrum is approximately given by the Planck formula, with a peak at this energy. Since the peak is so close to the red end of the visible spectrum, much of the sun's energy is emitted as visible light. (If you've learned elsewhere that the sun's spectrum peaks in the middle of the visible spectrum at about 500 nm, and you're worried about the discrepancy, go back and work Problem 7.39.)

A tiny fraction of the sun's radiation is absorbed by the earth, warming the earth's surface to a temperature suitable for life. But the earth doesn't just keep getting hotter and hotter; it also *emits* radiation into space, at the same rate, on average. This balance between absorption and emission gives us a way to estimate the earth's equilibrium surface temperature.

As a first crude estimate, let's pretend that the earth is a perfect blackbody at all wavelengths. Then the power absorbed is the solar constant times the earth's cross-sectional area as viewed from the sun, $\pi R^2$. The power emitted, meanwhile, is given by Stefan's law, with $A$ being the full surface area of the earth, $4\pi R^2$, and $T$ being the effective average surface temperature. Setting the power absorbed equal to the power emitted gives

$$\begin{aligned} &(\text{solar constant}) \cdot \pi R^2 = 4\pi R^2 \sigma T^4 \\ \Rightarrow\ & T = \left(\frac{1370 \text{ W/m}^2}{4 \cdot 5.67 \times 10^{-8} \text{ W/m}^2{\cdot}\text{K}^4}\right)^{1/4} = 279 \text{ K}. \end{aligned} \tag{7.102}$$

This is extremely close to the measured average temperature of 288 K (15°C).

However, the earth is *not* a perfect blackbody. About 30% of the sunlight striking the earth is reflected directly back into space, mostly by clouds. Taking reflection into account brings the earth's predicted average temperature down to a frigid 255 K.

Since a poor absorber is also a poor emitter, you might think we could bring the earth's predicted temperature back up by taking the imperfect emissivity into account on the right-hand side of equation 7.102. Unfortunately, this doesn't work. There's no particular reason why the earth's emissivity should be the same for the infrared light emitted as for the visible light absorbed, and in fact, the earth's surface (like almost any nonmetal) is a very efficient emitter at infrared wavelengths. But there's another mechanism that saves us: Water vapor and carbon dioxide in earth's atmosphere make the atmosphere mostly opaque at wavelengths above a few microns, so if you look at the earth from space with an eye sensitive to infrared light, what you see is mostly the atmosphere, not the surface. The equilibrium temperature of 255 K applies (roughly) to the atmosphere, while the surface below is heated both by the incoming sunlight and by the atmospheric "blanket." If we model the atmosphere as a single layer that is transparent to visible light but opaque to infrared, we get the situation shown in Figure 7.25. Equilibrium requires that the energy of the incident sunlight (minus what is reflected) be equal to the energy emitted upward by the atmosphere, which in turn is equal to the energy radiated downward by the atmosphere. Therefore the earth's surface receives twice as much energy (in this simplified model) as it would from sunlight alone. According to equation 7.102, this mechanism raises the surface temperature by a factor of $2^{1/4}$, to 303 K. This is a bit high, but then, the atmosphere isn't just a single perfectly opaque layer. By the way, this mechanism is called the **greenhouse effect**, even though most greenhouses depend primarily on a different mechanism (namely, limiting convective cooling).

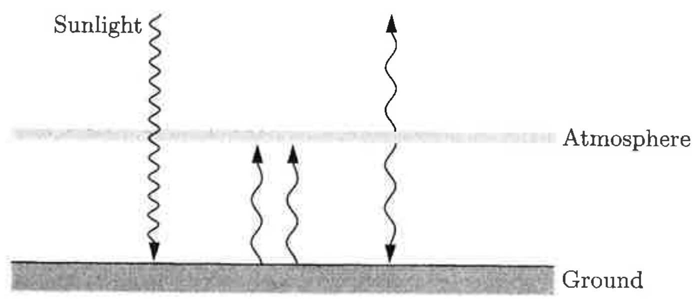


**Figure 7.25.** Earth's atmosphere is mostly transparent to incoming sunlight, but opaque to the infrared light radiated upward by earth's surface. If we model the atmosphere as a single layer, then equilibrium requires that earth's surface receive as much energy from the atmosphere as from the sun.

**Problem 7.54.** The sun is the only star whose size we can easily measure directly; astronomers therefore estimate the sizes of other stars using Stefan's law.

**(a)** The spectrum of Sirius A, plotted as a function of energy, peaks at a photon energy of 2.4 eV, while Sirius A is approximately 24 times as luminous as the sun. How does the radius of Sirius A compare to the sun's radius?

**(b)** Sirius B, the companion of Sirius A (see Figure 7.12), is only 3% as luminous as the sun. Its spectrum, plotted as a function of energy, peaks at about 7 eV. How does its radius compare to that of the sun?