Seminar 11
Problem Solutions

Problem 1

Mass of atoms...

i) Copper $\frac{m}{V} = 8.93 \frac{g}{cm^3}$

$m_A = 63.5$ g/mol

see next page for alternative *

Thus one mole has volume $\frac{m_A}{m/V} = 7.11 \times 10^{-6} m^3$

So if there is one conduction electron per atom of this mole: *

$$\frac{N}{V} = \frac{6.02 \times 10^{23}}{7.11 \times 10^{-6} m^3} = 8.47 \times 10^{28} \frac{\text{electrons}}{m^3}$$

$$\therefore \epsilon_F = \frac{h^2}{8m}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{N}{V}\right)^{2/3}$$

where $m = m_e = 9.11 \times 10^{-31}$ kg

$$= 1.13 \times 10^{-18} J$$

Pressure
$P = \frac{2}{5} \rho \epsilon_F = 3.8 \times 10^{10}$ Pa

Wow! over 10^5 atmospheres

Whereas $kT_{room} \cong 4.1 \times 10^{-21} J$

so $\epsilon_F \gg kT_{room}$. Yes, this is low enough to treat the electrons as a classical gas.

ii) Now spose $\epsilon = pc$. Want to rederive $\mu(T=0) = \epsilon_F$ ....

We still have particles in box ... same counting arguments (eg. Schroeder p. 294) yield

$$N = 2 \times \text{vol of } \tfrac{1}{8} \text{ sphere} = \frac{\pi n_{max}^3}{3}$$

Now $\epsilon_F = \hbar k_{max} c = \frac{hcn_{max}}{2L}$

$\therefore$

*

Copper's mass is $8.93\ g/cm^3 = 8.93 \times 10^3\ \frac{kg}{m^3}$

$M_{\text{atom of copper}} = 63.5 \cdot 1.67 \times 10^{-27}\ kg$

periodic table — m_p

-or- fact that
copper's mass is
given as $63.5\ g/mole$

$\therefore$ Copper has $\dfrac{8.93 \times 10^3\ kg/m^3}{63.5 \times 1.67 \times 10^{-27}\ kg/atom}$

$= 8.42 \times 10^{28}\ atoms/m^3$.

This is N/V

24

$$\therefore \quad \epsilon_F = \frac{hc}{2L}\left(\frac{3}{\pi}\right)^{1/3} N^{1/3}$$

$$\epsilon_F = hc\left(\frac{3N}{8\pi V}\right)^{1/3}$$

$$\left(\text{vs. } \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} \text{ for classical gas}\right)$$

Total energy

$$U = 2\iint_0^{4\pi/8} d\Omega \int_0^{n_{max}} dn\, n^2 \frac{hcn}{2L}$$

$$= \frac{\pi hc}{2L}\frac{n_{max}^4}{4} = \frac{\pi hc}{8L}\left(\frac{3N}{\pi}\right)^{4/3}$$

so

$$U = \frac{3^{4/3} hc}{8\pi^{1/3}} N \left(\frac{N}{V}\right)^{1/3}$$

$$U = \frac{3}{4} N \epsilon_F$$

(vs. $3/5 N \epsilon_F$ for classical case)

Problem 2

Crystalline Solids: Einstein + Debye Models

G & T Problem 6.35

(a) Find λ_D corresponding to the Debye frequency ω_D & show $\lambda_D \simeq a$.
(This justifies that must have $\omega < \omega_D$ ... atoms can't oscillate with $\lambda < a$)

Solution: Recall where ω_D comes from.

We have 3N degrees of freedom, so can only have 3N normal modes. Thus

$$3N = \int_0^{\omega_D} g(\omega)\, d\omega$$

Whence this?
A familiar mode-counting argument:

$$g(k)\,dk = V \frac{k^2\,dk}{2\pi^2} \text{ in 3d}$$

And $\omega = ck$.
Also, there are 3 modes at each k with different speeds c_ℓ + c_t so we define $\frac{3}{\bar{c}^3} = \frac{2}{c_t^3} + \frac{1}{c_\ell^3}$

with $g(\omega) = V \dfrac{3\omega^2}{2\pi^2 \bar{c}^3}$

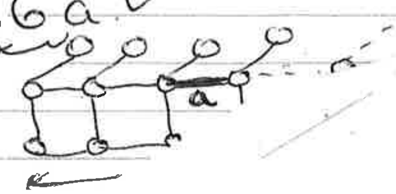
Doing this integral yields G+T Eq 6.200

$$\boxed{\omega_D = 2\pi \bar{c} \left(\frac{3\rho}{4\pi}\right)^{1/3}}$$

where $\rho = N/V$

Thus, $\lambda_D \simeq \dfrac{\bar{c}\,2\pi}{\omega_D} = \left(\dfrac{4\pi}{3}\right)^{1/3} \rho^{-1/3}$. If we imagine lattice is simple cubic, $V = L^3$ and

$$\rho^{-1/3} = \frac{L}{N^{1/3}} = \frac{N^{1/3} a}{N^{1/3}} = a \Rightarrow \lambda_D \simeq 1.6a$$



(b) Want to show energy in (6.202) is propto $\begin{cases} T & \text{for high } T \\ T^4 & \text{for low } T \end{cases}$

Sol'n:

$$E = 9Nk_BT\left(\frac{T}{T_D}\right)^3\int_0^{T_D/T}\frac{x^3}{e^x-1}dx \quad (6.202)$$

Spose T is small ... s.t. $T_D/T \equiv \infty$

Then integral is just a number* and

$$E \cong \frac{9\pi^4}{15}\frac{Nk_BT^4}{T_D^3} = \frac{3}{5}\frac{\pi^4 Nk_B}{T_D^3}T^4 \checkmark$$

Spose T is large ... Then x never gets very large in integrand and we can approximate $e^x - 1 \approx 1 + x - 1 \approx x$

$$\text{Thus}\quad E \cong 9Nk_BT\left(\frac{T}{T_D}\right)^3\int_0^{T_D/T} x^2\,dx$$

$$= 3Nk_BT\left(\frac{T}{T_D}\right)^3\left(\frac{T_D}{T}\right)^3$$

$$E \cong 3Nk_BT$$

* Actually, The number is discussed in B+B as a Bose Integral: $I_B(n) \equiv \zeta(n+1)\Gamma(n+1)$

Thus, integral is $\zeta(4)\Gamma(4) = \frac{\pi^4}{90}\cdot 3! = \frac{\pi^4}{15}$

(c) Plot T-dependence of E given by the two theories on same graph and compare.

Solution: I have to say it's a bit tough to choose how to pick T_E for this comparison. That is,

$T_E = \frac{\hbar \omega_E}{k}$ where ω_E is some given frequ. for each oscillator. In contrast,

$T_D = \frac{\hbar \omega_D}{k}$ where ω_D is set by
→ speed of sound $\bar{c}$
→ density $\frac{N}{V}$ of crystal

Easiest thing is I guess, just let $\omega_E \equiv \omega_D$. So we do this in what follows. It looks like this choice yields a systematic offset for larger T values ... so perhaps we would get an even better fit with ω_E a bit higher*...

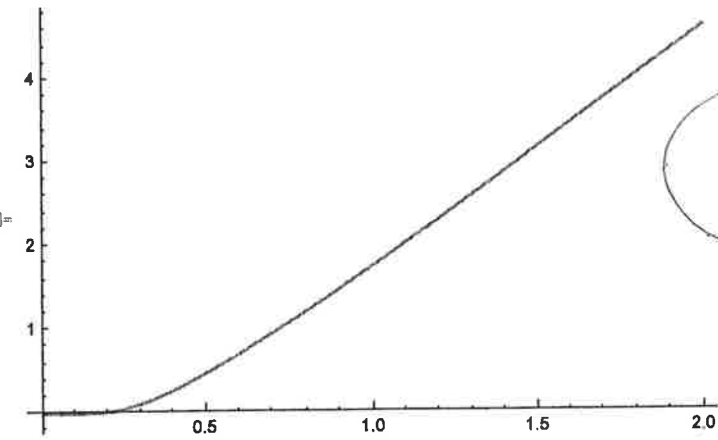
Still, slope at high T is 3Nk for both!

* which seems **odd** as ω_D is a high frequency cutoff so wouldn't we expect that ω_E being more like an average frequency in Debye crystal, hence $\omega_E < \omega_D$, would fit better

2-4

In[77]:= **EnergyEinstein = Plot[3 / (Exp[1 / x] - 1), {x, 0, 2}]**

Out[77]=



$$E_{Einstein} = \frac{3NkT_E}{e^{T_E/T}-1}$$

In[53]:= **x = .01**

Out[53]= 0.01

In[54]:= **9 * x^4 * NIntegrate[y^3 / (Exp[y] - 1), {y, 0, 1 / x}]**

Out[54]= 5.84455×10^{-7}

In[55]:= **x = .1**

Out[55]= 0.1

In[56]:= **9 * x^4 * NIntegrate[y^3 / (Exp[y] - 1), {y, 0, 1 / x}]**

Out[56]= 0.00578873

In[57]:= **x = .2**

Out[57]= 0.2

etc

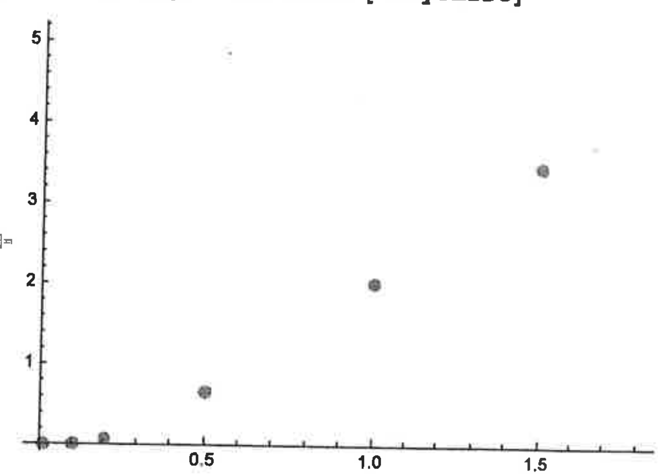
In[72]:= **DebyeList = {{0.01, 0}, {0.1, .006} , {0.2, .07},
{0.5, .66}, {1.0, 2.02}, {1.5, 3.47}, {2.0, 4.95}}**

Out[72]= {{0.01, 0}, {0.1, 0.006}, {0.2, 0.07}, {0.5, 0.66}, {1., 2.02}, {1.5, 3.47}, {2., 4.95}}

$$E_{Debye} = 9NkT_D\left(\frac{T}{T_D}\right)^4\int_0^{T_D/T}\frac{x^3}{e^x-1}\,dx$$

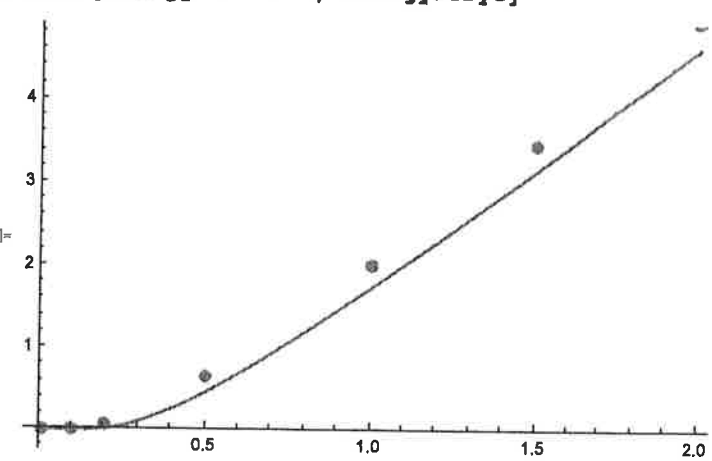
In[74]:= **EnergyDebye = ListPlot[DebyeList]**

Out[74]=



In[78]:= **Show[EnergyEinstein, EnergyDebye]**

Out[78]=



2-4a

Also note: B+B has comparison, not of Energies but of spec. heats ... also choosing $T_E = T_D$

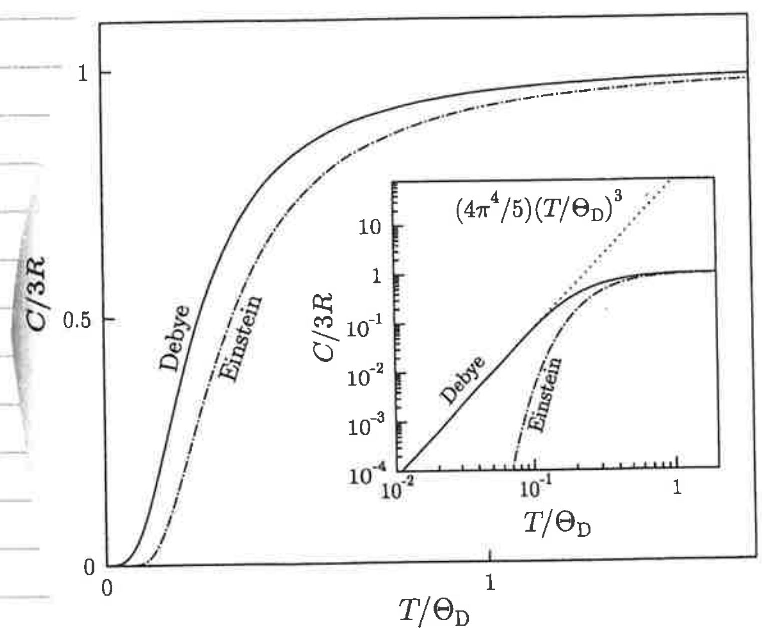


Fig. 24.3 The molar specific heat capacity for the Einstein solid and the Debye solid, according to eqn 24.8 and eqn 24.25 respectively. The inset shows the same information on a log–log scale, illustrating the difference between the low-temperature-specific heat capacities of the two models. The Debye model predicts a cubic temperature dependence at low temperature according to eqn 24.28, as shown by the dotted line. The figure is drawn with $\Theta_E = \Theta_D$.

(d) Derive expressions for energy analogous to (6.202) for 1d & 2d crystals. Thence find specific heats & how they depend on T.

$\boxed{1D}$ $g(k)\,dk = \frac{L}{\pi}\,dk$

$\Rightarrow g(\omega)\,d\omega = \frac{L}{\pi c}\,d\omega$

only one polariz'n

where $\rho = \frac{N}{L}$

So $N = \int_0^{\omega_D} \frac{L}{\pi c}\,d\omega = \frac{L\omega_D}{\pi c} \Rightarrow \omega_D = c\pi\rho$

Thus $E = \frac{L}{\pi c}\int_0^{kT_D/\hbar} \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\,d\omega$

Let $x = \beta\hbar\omega$, $dx = \beta\hbar\,d\omega$ $\Rightarrow$ $E = NkT\left(\frac{T}{T_D}\right)\int_0^{T_D/T} \frac{x}{e^x-1}\,dx$

Thus $E \sim \begin{cases} NkT & T \text{ high} \\ \frac{NkT^2}{T_D} & T \text{ low} \end{cases}$

Not sure about polariz'ns.... maybe there are two

$\boxed{2D}$ $g(\omega) = 2\,\frac{L^2\omega\,d\omega}{2\pi c^2}$

$\therefore$ $E = 4NkT\left(\frac{T}{T_D}\right)^2\int_0^{T_D/T} \frac{x^2}{e^x-1}\,dx$

$E \sim \begin{cases} 2NkT & T \text{ high} \\ 8NkT\left(\frac{T}{T_D}\right)^2 \sim T^3 & T \text{ low} \end{cases}$

?

1D $C_V \sim \begin{cases} Nk & T\,hi \\ T & T\,lo \end{cases}$

2D $C_V \sim \begin{cases} 2Nk & T\,hi \\ T^2 & T\,lo \end{cases}$

To help with G+T Prob 6.35 #2

extra-1

Let's remember how we do this useful thing:

DOS To find: $g(k)$ + $g(\epsilon)$ for 1d, 2d + 3d nonrel. particles. Also do so for photons ...

1d

Consider a 1d box

$$\psi(x) = \sqrt{\frac{2}{L}} \sin kx$$

$$k = \frac{n\pi}{L} \quad n = 0, 1, 2 \ldots$$

A single point in k space occupies length $\frac{\pi}{L}$. We will only allow +ve values of wave vector k ....

allowed states btwn k + k+dk is

$$\frac{dk}{\frac{\pi}{L}} = \frac{L\,dk}{\pi} = g(k)\,dk$$

Thus since $g(\epsilon)d\epsilon = g(k)\,dk$

$$\epsilon = \frac{\hbar^2 k^2}{2m} \Rightarrow k = \frac{\sqrt{2m\epsilon}}{\hbar}$$

$$d\epsilon = \frac{\cancel{2}\hbar^2 k\,dk}{\cancel{2}m} \quad , \quad \frac{dk}{d\epsilon} = \frac{m}{\hbar^2 k}$$

$$g(\epsilon)d\epsilon = \frac{L}{\pi}dk \Rightarrow g(\epsilon) = \frac{L}{\pi}\frac{dk}{d\epsilon}$$

$$g(\epsilon) = \frac{L}{\pi}\frac{m}{\hbar^2 k} = \frac{L}{\pi}\frac{m\,\hbar}{\hbar^2\sqrt{2m\epsilon}} = \frac{L\sqrt{m}}{\pi\hbar\sqrt{2}}\epsilon^{-1/2}$$

$p = \frac{h}{\lambda} = hc\nu$ $\quad p = \frac{h}{\lambda} = \frac{h\nu}{c}$

extra-2

easier to visualize

2d

$$\Gamma(k) = \frac{1}{4}\frac{\pi k^2}{(\pi/L)^2}$$

area

area per state

quadrant

$$\therefore \; g(k)\,dk = \frac{d\Gamma}{dk}\,dk = \underbrace{L^2 \frac{k}{2\pi}}_{g(k)}\,dk$$

$$\Rightarrow \quad g(\epsilon)\,d\epsilon = g(k)\,dk$$

$$\therefore \; g(\epsilon) = g(k)\,\frac{dk}{d\epsilon}$$

$$g(\epsilon) = L^2 \frac{k}{2\pi}\,\frac{m}{\hbar^2 k} = \frac{L^2 m}{2\pi\hbar^2}$$

3d

$$\Gamma(k) = \frac{1}{8}\frac{4\pi k^3/3}{(\pi/L)^3} \; ; \; g(k) = \frac{\partial\Gamma}{dk}$$

$$\Rightarrow \boxed{g(k)\,dk = L^3 \frac{k^2\,dk}{2\pi^2}}$$

$$\Rightarrow \; g(\epsilon) = g(k)\,\frac{dk}{d\epsilon}$$

$$= L^3 \frac{2m\epsilon}{2\pi^3\hbar^2}\;\frac{m}{\hbar^2}\frac{\hbar}{\sqrt{2m\epsilon}}$$

$$= L^3 \frac{m^{3/2}}{\sqrt{2}\,\pi^2\hbar^3}\,\epsilon^{1/2}$$

extra 3

For photons . . . would be that
$g(k)\,dk$ is just like for matter
save that $\exists$ 2 polariz'n
states. So

$$g(k)\,dk = \frac{2V\,k^2\,dk}{2\pi^2} = \frac{Vk^2\,dk}{\pi^2}.$$

where now we use $V = L^3$.

Since $\epsilon = \hbar\omega = \hbar c k$
$\Rightarrow d\epsilon = \hbar c\,dk$
we derive

$$g(\epsilon)\,d\epsilon = g(k)\,dk$$

$$g(\epsilon)\,d\epsilon = \frac{V\,\epsilon^2\,d\epsilon}{\pi^2\hbar^3 c^3}$$

Problem 3 Neutron Star
G+T Problem 6.55

Problem 6.55. A neutron star can be considered to be a collection of non-interacting neutrons, which are spin 1/2 fermions. A typical neutron star has a mass M close to one solar mass $M_\odot \approx 2 \times 10^{30}$ kg. The mass of a neutron is about $m = 1.67 \times 10^{-27}$ kg. In the following we will estimate the radius R of the neutron star.

(a) $T=0$ energy?

$$E = U = \tfrac{3}{5} N \epsilon_F = \tfrac{3}{5} N (3\pi^2)^{2/3} \frac{\hbar^2}{2m} \rho^{2/3}$$

Now $N = \dfrac{M}{m}$, $\rho = \dfrac{N}{V} = \left(\dfrac{M}{m}\right)\left(\dfrac{1}{\frac{4\pi}{3}R^3}\right)$

So

$$E = U = \frac{\hbar^2}{5R^2}\left(\frac{M^5 \pi^2 3^7}{m^8 2^7}\right)^{1/3}$$

(b) To show: $E_{grav} = -\dfrac{3GM^2}{5R}$

Answer: PE needed to bring shell of volume $4\pi r^2 \Delta r$ in from ∞ is

$$\Delta U_{grav} = -\frac{G\rho_{mass}^2 \left(\frac{4\pi}{3}r^3\right)\left(4\pi r^2 \Delta r\right)}{r}$$

$\rho_{mass} = \dfrac{Nm}{V}$

Thus

$$E_{grav} = \int_0^R dU(r) = -\frac{3GM^2}{5R}; \quad \text{where } M = \rho \frac{4\pi}{3} R^3$$

$$N = \frac{M}{m}$$

(c) Minimize $E_{grav.}$ wrt R:

We have two terms in energy.

Degeneracy energy $\propto +1/R^2$

Gravitational energy $\propto -1/R$

$$\frac{dE}{dR} = \frac{d}{dR}\left(\frac{A}{R^2} - \frac{B}{R}\right) = \frac{-2A}{R^3} + \frac{B}{R^2} = 0$$

$$\Rightarrow 2A = BR$$

$$\Rightarrow R = \frac{2A}{B} = \underline{\frac{2\hbar^2}{3G}\left(\frac{\pi^2 3^7}{M m^8 2^7}\right)^{1/3}}$$

(d) What is R in km?

I get about 12 km

$$\rho_{mass} = \frac{2\times10^{30}\text{ kg}}{\frac{4\pi}{3}(12\times10^3\text{ m})^3} \approx 3\times10^{17}\ \frac{\text{kg}}{\text{m}^3}$$

Wow! Water is 1×10^3 kg/m³. A bit denser!

(e) $$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2)^{2/3}\rho^{2/3} \qquad \left(\frac{N}{m}\right)^{2/3}$$

so $\underline{\epsilon_F = 9\times10^{-12}\text{ J}}$

Thus $T_F = \frac{\epsilon_F}{k} = \underline{6.5\times10^{11}\text{ K}}$

Higher than star's stated temperature of $T = 10^5$ K so "low temperature" approx. is valid.

(f) For an individual neutron, its rest energy is

$$\epsilon_0 = mc^2 = (1.67 \times 10^{-27}\ \mathrm{kg})(3 \times 10^8\ \tfrac{\mathrm{m}}{\mathrm{s}})^2$$

$$\epsilon_0 = 1.5 \times 10^{-10}\ \mathrm{J}$$

Recall from part (e) that $\epsilon_F = 9 \times 10^{-12}\ \mathrm{J}$. So yes, $\epsilon_0 \gg \epsilon_F$ and we can treat these neutrons nonrelativistically.

Exercise for the reader :) Go back & change ? to get relativistic neutron star?

Problem 4 → A fermi gas at $0 < T \ll T_F$
G & T problem 6.31 (a)-(c)

(a) Fill in missing steps in (6.163) - (6.174)

$\Delta E = E(T) - E(T=0)$ is ... from chapter

$$\int_0^\infty \epsilon\, \bar{n}(\epsilon) g(\epsilon) d\epsilon - \int_0^{\epsilon_F} \epsilon\, g(\epsilon)\, d\epsilon \qquad (6.163a)$$

$$\Delta E = \int_0^{\epsilon_F} \epsilon \left[\bar{n}(\epsilon) - 1 \right] g(\epsilon)\, d\epsilon + \int_{\epsilon_F}^\infty \epsilon\, \bar{n}(\epsilon) g(\epsilon) d\epsilon \qquad (6.163b)$$

Now use $N = \underbrace{\int_0^\infty \bar{n}(\epsilon)\, g(\epsilon)\, d\epsilon}_{\text{break up into 2 integrals}} = \int_0^{\epsilon_F} g(\epsilon) d\epsilon \qquad (6.164)$

Now × by ϵ_F +

$$\Rightarrow \int_0^{\epsilon_F} \epsilon_F\, \bar{n}(\epsilon) g(\epsilon)\, d\epsilon + \int_{\epsilon_F}^{\infty} \epsilon_F\, \bar{n}(\epsilon) g(\epsilon) d\epsilon = \int_0^{\epsilon_F} \epsilon_F\, g(\epsilon) d\epsilon$$

bring over to LHS

$$\Rightarrow \int_0^{\epsilon_F} \epsilon_F \left(\bar{n}(\epsilon) - 1\right) g(\epsilon)\, d\epsilon + \int_{\epsilon_F}^{\infty} \epsilon_F\, \bar{n}(\epsilon)\, g(\epsilon)\, d\epsilon = 0 \qquad (6.165b)$$

Now rewrite 6.163b by inserting this "0" from (6.165b)

$$\Delta E = \int_{\epsilon_F}^{\infty} (\epsilon - \epsilon_F)\, \bar{n}(\epsilon)\, g(\epsilon)\, d\epsilon + \int_0^{\epsilon_F} (\epsilon_F - \epsilon) \left[1 - \bar{n}(\epsilon) \right] g(\epsilon) d\epsilon$$

Need $C_V = \frac{d\,\Delta E}{dT}$. Since $\bar{n}(\epsilon)$ is only place where a T dependence comes in,

$$C_V = \int_0^\infty (\epsilon - \epsilon_F)\frac{d\bar{n}(\epsilon)}{dT} g(\epsilon)\, d\epsilon \qquad (6.167)$$

We are going to take $g(\epsilon) = g(\epsilon_F)$ and pull it from $\int$ (This seems sketchy...) but it is b/c $\frac{d\bar{n}(\epsilon)}{dT} \approx 0$ except right around $\epsilon = \epsilon_F$.

$$\therefore\ C_V = g(\epsilon_F)\int_0^\infty (\epsilon - \epsilon_F)\frac{d\bar{n}(\epsilon)}{dT}\, d\epsilon \qquad (6.168)$$

We are also going to set $\mu(T) = \epsilon_F$. Then

$$\frac{d\bar{n}}{dT} = \frac{d\bar{n}}{d\beta}\frac{d\beta}{dT} = \frac{1}{kT^2}\frac{(\epsilon - \epsilon_F)\, e^{\beta(\epsilon - \epsilon_F)}}{\left[e^{\beta(\epsilon-\epsilon_F)} + 1\right]^2} \qquad (6.169$$

$x \equiv \frac{(\epsilon - \epsilon_F)}{kT}$

$\Rightarrow d\epsilon = kT\,dx$

$$C_V = k^2 T g(\epsilon_F)\int_{-\beta\epsilon_F}^\infty \frac{x^2 e^x}{(e^x + 1)^2}\, dx \qquad (6.170$$

$\epsilon = 0 \Rightarrow x = -\frac{\epsilon_F}{kT}$

e^x gets very small below $x = -\beta\epsilon_F$ since $T \ll T_F$ so rewrite as

$$C_V = k^2 T g(\epsilon_F)\int_{-\infty}^\infty \frac{x^2 e^x}{(e^x + 1)^2}\, dx \qquad (6.171)$$

$(1+x)^{5/2} \simeq 1 + 5/2\, x$

Numerically that Sol is $\frac{\pi^2}{3}$, thus

$$C_v = k^2 T g(\epsilon_F) \frac{\pi^2}{3} \qquad (6.172)$$

Now (finally an actual missing step!)

WTS $\quad g(\epsilon_F) = \frac{3N}{2\epsilon_F} = \frac{3N}{2kT_F}$

proof
$$g(\epsilon) = A\epsilon^{1/2}$$
$$g(\epsilon_F) = g(\epsilon_F)$$
$$\therefore A = g(\epsilon_F)\,\epsilon_F^{-1/2}$$

Slick!

(could also plug in all #'s). Thus $N = \int_0^{\epsilon_F} A\epsilon^{1/2} d\epsilon = g(\epsilon_F) \int_0^{\epsilon_F} \left(\frac{\epsilon}{\epsilon_F}\right)^{1/2} d\epsilon$;

$$N = g(\epsilon_F)\, 2/3\, \epsilon_F^{3/2} / \epsilon_F^{1/2} = g(\epsilon_F)\, 2/3\, \epsilon_F$$

$$\therefore g(\epsilon_F) = \frac{3}{2} N/\epsilon_F \qquad (6.173)$$

Finally! $\quad C_v = \frac{\pi^2}{2} Nk \frac{T}{T_F}$

(b) To do: Use (6.175) and (6.179)

to show $\quad P = 2/5\, \rho\, \epsilon_F \left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right]$

Answer: Here are these eq's

$$\Omega = -\frac{2}{3}\frac{2^{1/2} V m^{3/2}}{\pi^2 \hbar^3}\left[\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}(kT)^2 \mu^{1/2}\right] \qquad (6.175)$$

$$\mu(T) = \epsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] \qquad (6.179)$$

So, lets sub (6.179) into (6.175). Also, Recall $\Omega = -PV$.

Thus $-\dfrac{\Omega}{V} = P = \dfrac{2}{3}\dfrac{2^{1/2}m^{3/2}}{\pi^2\hbar^3}\left[\dfrac{2}{5}\left(\epsilon_F\left[1-\dfrac{\pi^2}{12}\left(\dfrac{T}{T_F}\right)^2\right]\right)^{5/2} +\right.$

$$\left. + \frac{\pi^2}{4}(kT)^2\left(\epsilon_F\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]\right)^{1/2}\right]$$

We want to only expand to order $\left(\frac{T}{T_F}\right)^2$.

We are also fishing for a factor of "ρ" in the expression for P that we must find. For this, use

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2)^{2/3}\rho^{2/3}$$

$$\Rightarrow \rho = \epsilon_F^{3/2}(3\pi^2)^{-1}\left(\frac{\hbar^2}{2m}\right)^{-3/2}$$

Putting it all together; the second term loses its second term:

$$P = \frac{\rho}{\epsilon_F^{3/2}}\left[\frac{2}{5}\left(\epsilon_F^{5/2}\left[1-\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]\right)^{5/2} + \right.$$

$$\left. + \frac{\pi^2}{4}\epsilon_F^{5/2}\left(\frac{T}{T_F}\right)^2 + \cancel{\phi}\right]$$

and we do a binomial expansion of the first term:

$$P = \frac{\rho}{\epsilon_F^{3/2}}\left[\frac{2}{5}\left(\epsilon_F^{5/2}\left[1-\frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2 + \frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right]\right)\right]$$

$$P = \rho\epsilon_F\frac{2}{5}\left[1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$$ yay!

(c) Want to show from general rel'n of E + PV

$$E = \tfrac{3}{5} N \epsilon_F \left[1 + \tfrac{5}{12}\pi^2 \left(\tfrac{T}{T_F}\right)^2 + \cdots \right]$$

This is easy. Just use $P = \tfrac{2}{3}\tfrac{E}{V}$

or $E = \tfrac{3}{2} PV = \tfrac{3}{2} PN/\rho$

(b) $\Rightarrow$ ✓ answer

(d) $$-\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu} = S$$

WTS $$S = \frac{\pi^2}{2} Nk \frac{T}{T_F}$$

(Evan got $\frac{\pi^2}{3} Nk \frac{T}{T_F}$)

6.175 is $$\Omega = -\frac{2}{3} \frac{2^{\frac{1}{2}} V m^{3/2}}{\pi^2 \hbar^3} \left[\tfrac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}(kT)^2 \mu^{1/2} \right]$$

Thus $$\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu} = -\frac{2}{3}\frac{2^{1/2} V m^{3/2}}{\pi^2 \hbar^3} \frac{2\pi^2}{4} k^2 T \mu^{1/2}$$

$=$

Now,

$$\epsilon_F = \frac{3^{2/3}\,\pi^{4/3}\,\hbar^2}{2m}\left(\frac{N}{V}\right)^{2/3} \qquad \text{Eq. } (6.178b)$$

Thus

$$-\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu} = \frac{N}{\epsilon_F^{3/2}}\,\frac{\pi^2}{2}k^2 T \mu^{1/2}$$

if we set $\mu = \epsilon_F$

$$S = kN\frac{\pi^2}{2\epsilon_F}(kT)$$

$$= \frac{kN\pi^2}{2}\left(\frac{T}{T_F}\right)$$

Text

Problem 6

Schroeder

Problem 6.57. Toy systems of fermions.

(a) Consider a system of noninteracting (spinless) fermions such that each particle can be a single particle state with energy 0, Δ, and 2Δ. Find an expression for Z_G using (6.224). Determine how the mean number of particles depends on μ for $T = 0$, $kT = \Delta/2$, and $kT = \Delta$.

Solution. Equation (6.224) is $Z_G = \sum_{N=1}^{\infty} e^{\beta\mu N} Z_N$. Because each fermion can be in only one of three microstates, the maximum value of N is 3. We define $y = e^{-\beta\Delta}$ and write

$$Z_1 = 1 + y + y^2 \tag{S6.232a}$$
$$Z_2 = y + y^2 + y^3 \tag{S6.232b}$$
$$Z_3 = y^3, \tag{S6.232c}$$

where the occupancy for the state $(0, \Delta, 2\Delta)$ is $(1,0,0)$, $(0,1,0)$ for one particle, $(0,0,1)$; $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ for two particles; and $(1,1,1)$ for three particles. The grand partition function is thus

$$Z_G = (1 + y + y^2)z + (y + y^2 + y^3)z^2 + y^3 z^3, \tag{S6.233}$$

where $z = e^{-\beta\mu}$. The mean number of particles is given by

$$\overline{N} = \frac{(1 + y + y^2)z + 2(y + y^2 + y^3)z^2 + 3y^3 z^3}{(1 + y + y^2)z + (y + y^2 + y^3)z^2 + y^3 z^3}. \tag{S6.234}$$

For $T = 0$, $y = 0$ and thus $\overline{N} = 1$. For $kT = \Delta/2$, $y = e^{-2} = 0.1353$, and the mean number of particles is given by

$$\overline{N} = \frac{1.15z + 0.312z^2 + 0.0074z^3}{1.15z + 0.156z^2 + 0.0025z^3} \approx \frac{1 + (0.312/1.15)z}{1 + (0.156/1.15)z} \tag{S6.235a}$$
$$\approx 1 + 0.136z = 1 + 0.136e^{2\mu/\Delta}, \tag{S6.235b}$$

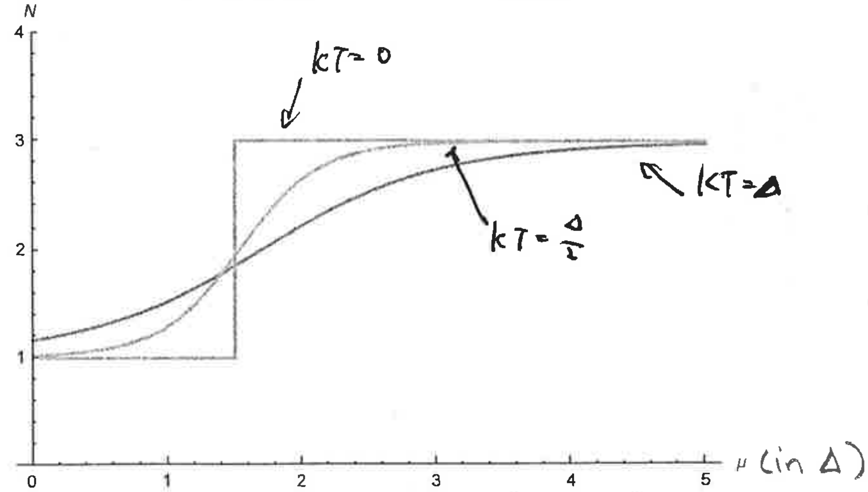
where we have assumed that $z \ll 1$, and used the fact that $(1+\delta)^{-1} \approx 1 - \delta$ for $\delta \ll 1$. For $kT = \Delta$, $y = e^{-1} = 0.3679$, and the mean number of particles is given by

$$\overline{N} = \frac{1.50z + 1.106z^2 + 0.149z^3}{1.50z + 0.553z^2 + 0.050z^3} \tag{S6.236a}$$
$$\approx \frac{1 + (1.106/1.50)z}{1 + (0.553/1.50)z} \approx 1 + 0.369z = 1 + 0.369e^{\mu/\Delta}. \tag{S6.236b}$$

Suppose z is not $\ll 1$. Here is plot

(thank you ZZ!)

Number of particles vs. Chemical potential



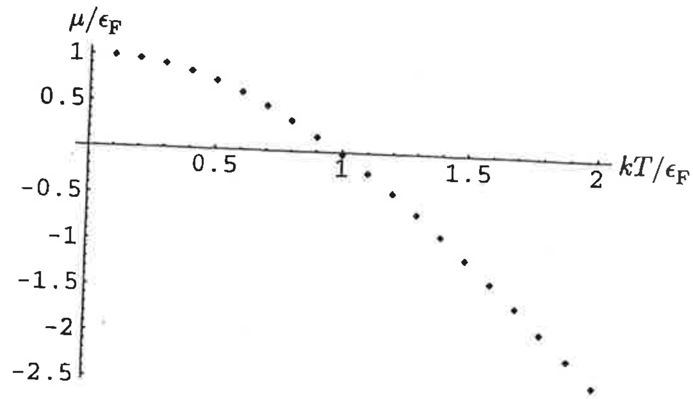
Then, for example, to find the actual value of c when $t = 1$, I typed simply:

```
FindRoot[1 == fermiN[c,1], {c,-1,0}]
```

(The numbers -1 and 0 specify a range of c values to use as initial trials in searching for a solution. The precise numbers used are not critical.) Although *Mathematica* complained with a warning message at this instruction, the answer it gave was quite reasonable: $c = -0.0215$. After this success, I got ambitious and asked for a table of solutions at 20 different t values:

```
mutable = Table[{t,FindRoot[1 == fermiN[c,t],{c,-1,0}][[1,2]]},{t,.1,2,.1}]
```

(Some of the details in this instruction are important only because I wanted to plot the table without retyping it. What it actually produces is a list of ordered pairs (t, c). The "[[1,2]]" after the FindRoot function strips off some unwanted stuff that would have interfered with plotting.) This time I got several warning messages but still plausible results, so I plotted it with the instruction `ListPlot[mutable]`, which produced the following graph:



(c) In principle, the energy calculation is actually easier than the μ calculation, but getting the previously calculated μ values into the *Mathematica* formula can be a bit tricky. I did it by defining an "interpolating function":

```
mu = Interpolation[mutable]
```

The function `mu[t]` can now be used to calculate μ (actually $c = \mu/\epsilon_F$) at any temperature. (Figure 7.16 was generated by typing `Plot[mu[t],{t,0,2}]`.) As for the energy integral itself, with the same set of substitutions it becomes

$$\frac{U}{N\epsilon_F} = \frac{3}{2}\int_0^\infty \frac{x^{3/2}}{e^{(x-c)/t}+1}\,dx,$$

which I programmed by typing

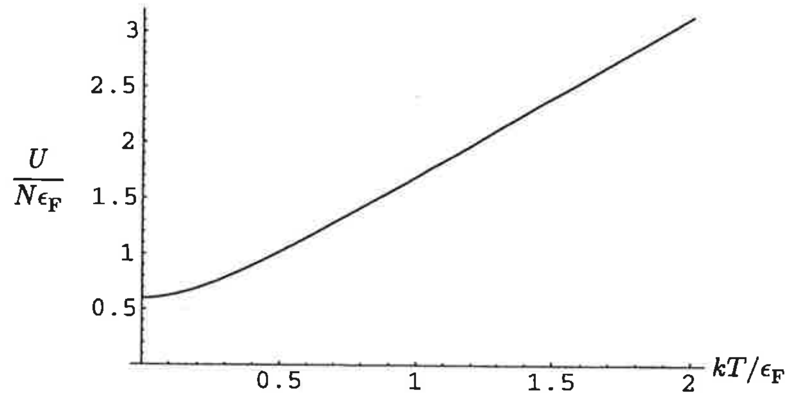
```
energy[t_] := 1.5*NIntegrate[x^1.5/(Exp[(x-mu[t])/t]+1),{x,0,Infinity}]
```

5-3

To plot it I gave the instruction

```
Plot[energy[t],{t,.01,2},PlotRange->{All,{0,3.2}}];
```

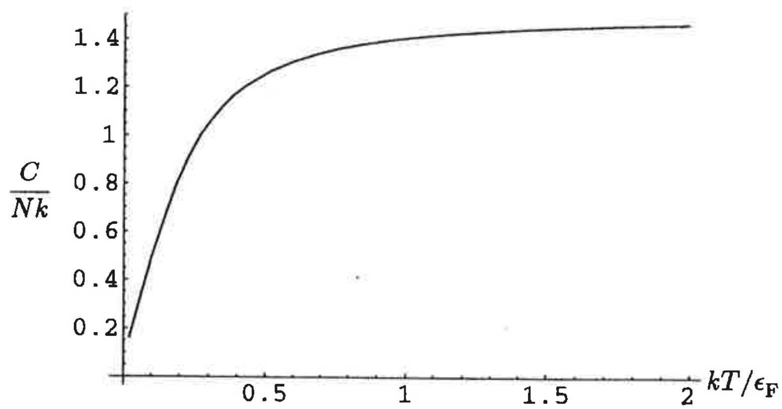
This produced the following graph:



Notice that $U/N\epsilon_F$ goes to 3/5 at $T = 0$, as expected. To obtain the heat capacity I numerically differentiated the energy function and then plotted the result:

```
heatcap[t_] := (energy[t+.01] - energy[t-.01])/.02
Plot[heatcap[t],{t,.02,1.99}];
```

(Not very elegant, but it does the job.) Note that this "heat capacity" function really calculates C/Nk, since the energy function is really $U/N\epsilon_F$ and t is in units of ϵ_F/k. Here's the plot:



At temperatures much less than ϵ_F/k, the heat capacity is approximately linear in T, as derived in the text. A temperatures much greater than ϵ_F/k, the heat capacity approaches $\frac{3}{2}Nk$, the value for an ordinary "monatomic" ideal gas.

6-1

Problem 6
Schroeder

Problem 6.57. Toy systems of fermions.

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$$Z_G = (1 + y + y^2)z + (y + y^2 + y^3)z^2 + y^3 z^3, \tag{S6.233}$$

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$$\approx \frac{1 + (1.106/1.50)z}{1 + (0.553/1.50)z} \approx 1 + 0.369z = 1 + 0.369e^{\mu/\Delta}. \tag{S6.236b}$$

Problem 6 cont
Schroeder 6.57 (b)
Given N identical non-interacting fermions with 2N distinct single particle states *. Spose

$\varepsilon = 2\Delta$ — — — — $\frac{2N}{3}$ states
$\varepsilon = \Delta$ — — — — $2N/_3$ states
$\varepsilon = 0$ — — — — $\frac{2N}{3}$ states

Want to show μ is indep of T
(Will find μ too...) Want to calculate T dependence of Energy & Heat Capacity.

Solution:

$$\bar{n}_s = \frac{1}{e^{\beta(\epsilon_s - \mu)} + 1}$$

$$\therefore\ N = \frac{2N}{3}\left[\frac{1}{e^{-\beta\mu}+1} + \frac{1}{e^{\beta(\Delta-\mu)}+1} + \frac{1}{e^{\beta(2\Delta-\mu)}+1}\right] \quad (1$$

Lets look at $\beta = 0$ ...

$$N = \frac{2N}{3}\left[\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right] = N \checkmark$$

* see next page for comment

Comment → This toy model is rather special. If we had not 2N states but an arbitrary number... call it A states, then

$$\bar{N} = \frac{A}{3}\left[\frac{1}{e^{-\beta\mu}+1} + \frac{1}{e^{\beta(\Delta-\mu)}+1} + \frac{1}{e^{\beta(2\Delta-\mu)}+1}\right)$$

For $\beta=0$, we'd have $\bar{N} = \frac{A}{2}$

If $\mu=\Delta$, we'd also have $\bar{N} = \frac{A}{2}$ (Do the algebra... it works out!)

But if $A \neq 2N$, $\mu \neq \Delta$

proof common denom

$$\frac{A}{3}\left[\frac{e^{\beta\Delta}+1}{(e^{-\beta\Delta}+1)(e^{\beta\Delta}+1)} + \frac{1}{2} + \frac{e^{-\beta\Delta}+1}{(e^{-\beta\Delta}+1)(e^{\beta\Delta}+1)}\right]$$

$$= \frac{A}{3}\left[\frac{e^{\beta\Delta}+e^{-\beta\Delta}+2}{e^{\beta\Delta}+e^{-\beta\Delta}+2} + \frac{1}{2}\right] = \frac{A}{3}\frac{3}{2} = \frac{A}{2}$$

We will show μ is indep of T by solving for $\mu(T)$ $\forall$ T.

Try $\mu = 0$ ... Nope ...

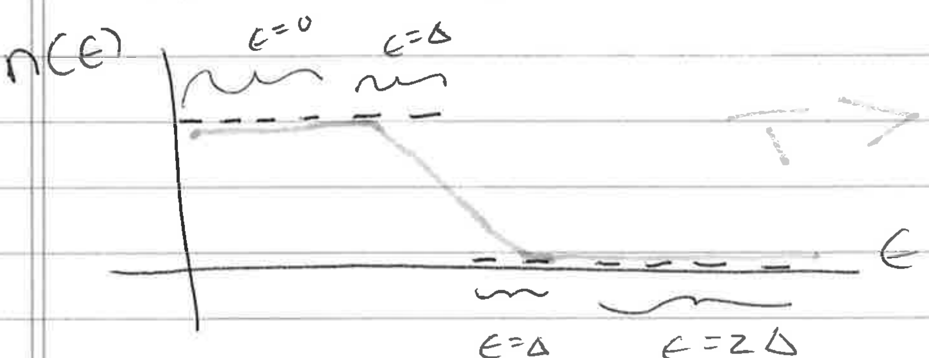
B/C this leads to N which varies with β and we do not want it to.

Try again ... $\mu = \Delta$. Bingo!

(as on p. 6-3) $\bar{N} = \frac{2N}{2} = N$ ✓

No matter what β is, this $\mu(T)$ works & gives us $\bar{N} = N$.

Here is picture



There is no <u>need</u> for μ to shift with T. Symmetry of # particles + # states ⇒ distribution with μ = const works $\forall$ T.

Now

$$E = \frac{2N\Delta}{3}(\bar{n}_1 + 2\bar{n}_2) = \frac{2N\Delta}{3}\left[\frac{1}{2} + \frac{2}{e^{\beta\Delta}+1}\right]$$

At $T=0$, $E = \frac{N\Delta}{3}$

As $T \to \infty$, $E \to \frac{2N\Delta}{3}\left[\frac{1}{2} + 2\cdot\frac{1}{2}\right]$

$$= N\Delta$$

These two values connect with a smooth curve.

Heat Capacity is

$$C = \frac{dE}{dt} = -\frac{1}{kT^2}\frac{dE}{d\beta}$$

$$= \frac{2N\Delta}{3kT^2}\left[\frac{2\Delta e^{\beta\Delta}}{(e^{\beta\Delta}+1)^2}\right]$$

$$= \frac{4N\Delta^2}{3kT^2}\left[\frac{e^{\beta\Delta}}{(e^{\beta\Delta}+1)^2}\right]$$

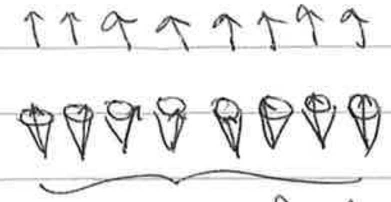
So as $T \to 0$, $C \to 0$

also as $T \to \infty$, $C \to 0$ ...

Peak in middle near $kT \simeq \Delta/2$

Problem 7

Schroeder 7.64 Magnons



one wavelength

ground state: $M = 2\mu_B N \equiv M_0$ (two electrons; number of atoms)

$$M = M_0 - 2\mu_B$$

$$f \propto \left(\frac{1}{\lambda}\right)^2 \quad \text{for } \lambda \text{ long} \ldots$$

$$\therefore \quad \epsilon = hf \;, \quad p = h/\lambda$$

$\therefore \; \epsilon \propto p^2$ like a particle

Can write $\epsilon = p^2/2m^*$ effective mass ...

Iron has $m^* = 1.24 \times 10^{-29}$ kg

Magnons have only 1 polariz'n ...

(a) To Show: At low T, # magnons/unit vol in 3d ferromagnet is

$$\frac{N_m}{V} = 2\pi\left(\frac{2m^*kT}{h^2}\right)^{3/2} \underbrace{\int_0^\infty \frac{\sqrt{x}}{e^x - 1}\,dx}_{\text{find numerically}}$$

This is going to be very much like density of bosonic particles when $\mu = 0$. That is

$$N_m = \int_0^\infty g(k)\bar{n}_k\,dk = \int_0^\infty \frac{4\pi k^2}{\left(\frac{2\pi}{L}\right)^3}\left(\frac{1}{e^{\frac{\beta\hbar^2k^2}{2m^*}} - 1}\right)dk$$

7-2

where as you see we've used B-E Statistics + also admitted only 1 polarizn state.

$$\text{Thus } N_m = V \frac{1}{2\pi^2} \int_0^\infty \frac{k^2}{e^{\beta \hbar^2 k^2 / 2m^*} - 1}\, dk$$

Familiar stuff!

Let $x = \frac{\beta \hbar^2 k^2}{2m^*} \Rightarrow dx = \frac{\beta \hbar^2 k\, dk}{m^*}$

$$\Rightarrow \frac{N_m}{V} = \frac{1}{2\pi^2} \frac{2^{1/2}}{\hbar^3} (m^* k_B T)^{3/2} \int_0^\infty \frac{x^{1/2}}{e^x - 1}\, dx$$

$$\text{Thus } \frac{N_m}{V} = 2\pi \left(\frac{2m^* k_B T}{h^2} \right)^{3/2} \int_0^\infty \frac{\sqrt{x}}{e^x - 1}\, dx$$

As required ....

Now, we do know about this Bose integral

It is $\zeta(3/2)\Gamma(3/2) \approx 2.315$

(b) To find: $\frac{M(0) - M(T)}{M(0)} \equiv \frac{\Delta M(T)}{M(0)}$

in form of $\left(\frac{T}{T_0}\right)^{3/2}$ + estimate T_0 for Iron ....

$\frac{\Delta M(T)}{M(0)}$ is $\frac{2\mu_B N_m}{2\mu_B N} = \frac{N_m}{N}$ which from

(a) is $2\pi \frac{V}{N} \left(2m^* k_B T / h^2\right)^{3/2} (2.315)$

Thus if we define

$$\left(\frac{1}{T_0}\right)^{3/2} = 41.4\,\frac{V}{N}\left(\frac{m^* k_B}{h^2}\right)^{3/2}$$

Then $\frac{\Delta m}{m_0} = \left(\frac{T}{T_0}\right)^{3/2}$

And $T_0 = (0.084)\left(\frac{N}{V}\right)^{2/3}\left(\frac{h^2}{m^* k_B}\right) K$

For iron, $\frac{N}{V} = \frac{N_A}{\text{moles/kg}} \times \frac{\text{kg}}{m^3}$

so $$T_0 = (0.084)\left[\frac{(6.02\times10^{23})(7844\ \text{kg/m}^3)}{0.0559\ \text{moles/kg}}\right]^{2/3} \times \left(\frac{(6.63\times10^{-34})^2\ J^2 s^2}{(1.24\times10^{-28})\ \text{kg}\ (1.38\times10^{-23}\ J/K)}\right)$$

or $\underline{T_0 = 4170\ K}$

(C) To find the Heat Capacity we need to find an expression for U this is given by

$$U = \int_0^\infty g(k)\,\epsilon_k \bar{n}_k\,dk$$

$$U = V \frac{1}{2\pi^2} \frac{\hbar^2}{8\pi^2 m^*} \int_0^\infty \frac{k^4}{e^{\frac{\beta\hbar^2 k^2}{8\pi^2 m^*}} - 1}\,dk$$

Again, letting $x = \frac{\beta\hbar^2 k^2}{2m^*}$ gives

$$U = \frac{V\hbar^2}{16\pi^4 m^*} \int_0^\infty \frac{k^3}{e^x - 1}\left(\frac{4\pi^2 m^*}{\beta\hbar^2}\right)dx, \quad k^3 = \left(\frac{8\pi^2 m^*}{\beta\hbar^2}\right)^{3/2} x^{3/2}$$

So

$$U = \frac{V\hbar^2}{16\pi^4 m^*}\left(\frac{4\pi^2 m^*}{\beta\hbar^2}\right)\left(\frac{8\pi^2 m^*}{\beta\hbar^2}\right)^{3/2} \int_0^\infty \frac{x^{3/2}}{e^x - 1}\,dx$$

$$U = (17.77)\,V \frac{m^{*3/2}}{h^3}(k_B T)^{5/2} \int_0^\infty \frac{x^{3/2}}{e^x - 1}\,dx$$

> Mathematica gives 1.78329

$$U = 31.7\,V \frac{m^{*3/2} k_B^{5/2}}{h^3} T^{5/2}$$

So

$$\boxed{C_V = \left.\frac{\partial U}{\partial T}\right|_V = 79.2\,V \frac{m^{*3/2} k_B^{5/2} T^{3/2}}{h^3}}$$

$$\text{So } \frac{C_V}{N k_B} = 79.2\left(\frac{V}{N}\right)\left(\frac{m^* k_B T}{h^2}\right)^{3/2} \equiv \left(\frac{T}{T_1}\right)^{3/2}$$

That is... Again, write as a ratio of T to effective temperature T_1 ....

$$T_1 = (0.054)\left(\frac{N}{V}\right)^{2/3}\left(\frac{h^2}{m^* k_B T}\right)$$

which reminds us of T_0 ....

in fact $T_1 = \left(\frac{0.054}{0.084}\right) T_0 = \underline{2680\,K}$

Compare... for phonons

$$\frac{C_V}{Nk_B} = \frac{12\pi^4}{5}\left(\frac{T}{T_D}\right)^3 \qquad (\text{Eq. } 7.115)$$

with $T_D = 470\,K$.

Thus

$$\frac{C_V}{Nk_B} = \begin{cases} T^3/444{,}000 & \text{for phonons} \\ T^{3/2}/139{,}000 & \text{for magnons} \end{cases} \qquad \text{with } T \text{ in kelvin}$$

(d.) In two dimensions, the density of states [illegible] is found by considering an annulus in k space. Thus, it will be proportional to $k\,dk$ not $k^2 dk$.

The number of magnons will now be given by

$$N_m \propto \int_0^\infty \frac{k}{e^{\beta\hbar^2 k^2/2m^*} - 1}\, dk .$$

Once again, let $x = \dfrac{\beta\hbar^2 k^2}{2m^*}$, $dx = \dfrac{\beta\hbar^2 k}{m^*}\, dk$, and we find

$$N_m \propto \int_0^\infty \frac{1}{e^x - 1}\, dx$$

Mathematica gives that this integral is equal to

$$\int_0^\infty \frac{1}{e^x - 1}\, dx = -x + \ln(1 - e^x)$$ Which <u>diverges</u>

over the limits of integration.