1-0

Seminar 10

Solutions

Problem 1

intro?

Problem 7.15. For a system of particles obeying the Boltzmann distribution, the total number of particles should be

$$N = \sum_{\text{all } s} \overline{n}_{\text{Boltzmann}} = \sum_s e^{-(\epsilon_s - \mu)/kT} = e^{\mu/kT} \sum_s e^{-\epsilon_s/kT}.$$

intro

But the sum in the last expression is just the single-particle partition function, Z_1, and therefore,

$$\frac{N}{Z_1} = e^{\mu/kT} \qquad \text{or} \qquad \mu = kT \ln \frac{N}{Z_1} = -kT \ln \frac{Z_1}{N}.$$

(I prefer to write $-\ln(Z_1/N)$ rather than $\ln(N/Z_1)$, since $Z_1 \gg N$ whenever the Boltzmann distribution applies.)

1

Problem 1) Finding μ from N: Bosons G&T 7.17

Starts out like prob. 7.16 where we have N identical fermions... Now, N identical bosons.

Have $q = 0, 1, \ldots 6$ as different macrostates. This is # quanta of energy: $E = q\eta$. Energy levels are equispaced:

$\vdots$

$\epsilon_2 = 2\eta$

$\epsilon_1 = \eta$

$\epsilon_0 = 0$

(a) Represent # Bosons in each level for each of these seven macrostates. With N Bosons, if some are in non-ϵ_0 state, rest are in ϵ_0. Will not write these all the way, but start out

	q=0	q=1	q=2		q=3		
	0	0	0	0	0	0	0
	0	0	0	0	0	0	1
$\epsilon_2 = 2\eta$	0	0	1	0	0	1	0
$\epsilon_1 = \eta$	0	1	0	2	3	1	0
$\epsilon_0 = 0$	N	N-1	N-1	N-2	N-3	N-2	N-1

	q=4				
$\epsilon_6 = 6\eta$	0	0	0	0	0
	0	0	0	0	0
	0	0	0	0	1
	0	0	0	1	0
	0	1	2	0	0
$\epsilon_1 = \eta$	4	2	0	1	0
$\epsilon_0 = 0$	~~N~~				

q=5						
0	0	0	0	0	0	0
0	0	0	0	0	0	1
0	0	0	0	0	1	0
0	0	0	1	1	0	0
0	1	2	0	1	0	0
5	3	1	2	0	1	0

stop writing ϵ_0 #'s

$q = 6$

$i=6$	0	0	0	0	0	0	0	0	0	0	1
.	0	0	0	0	0	0	0	0	0	1	0
.	0	0	0	0	0	0	0	1	1	0	0
.	0	0	0	0	2	1	1	0	0	0	0
$i=2$	0	1	2	3	0	1	0	1	0	0	0
$i=1$	6	4	2	0	0	1	3	0	2	1	0

(b) Want $\overline{n}_i$ for $q=6$

This means summing across horizontal row. The occupancy is technically $\sum_n n P(n)$ (7.22)

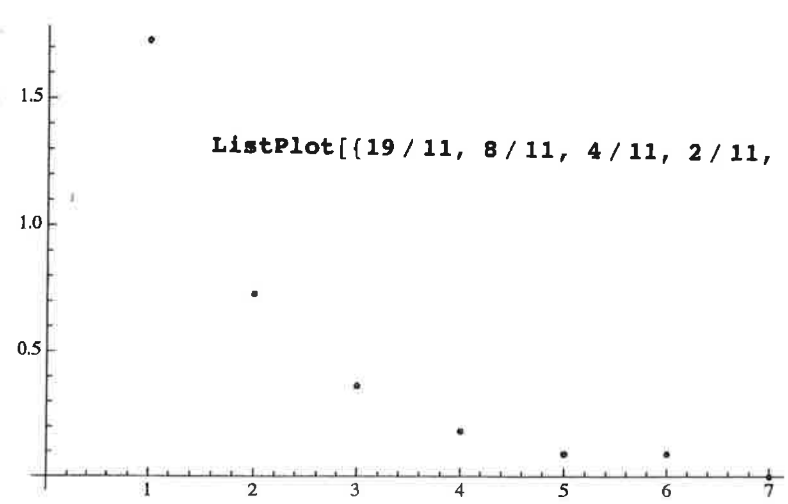
Since there are 11 states, $P(n) = \frac{1}{11}$ $\forall$ n.

$\overline{n}_{i=1} = 19/11$, $\overline{n}_2 = \frac{8}{11}$, $\overline{n}_3 = \frac{4}{11}$,

$\overline{n}_4 = 2/11$, $\overline{n}_5 = 1/11$, $\overline{n}_6 = \frac{1}{11}$, $\overline{n}_7 = 0, \ldots$

and $\overline{n}_0 = N - \sum_{i=1}^{\infty} nP(n) = \underline{N - 35/11}$

(c)



(c) What μ + T would you need to fit graph?

μ ~~we know~~ a priori ≈ 0 b/c we have $\bar{n}_{\epsilon=\mu} \approx \infty$.

So we just need to fit data to

$$\bar{n}_{BE} = \frac{1}{e^{q/kT} - 1}$$

Get $T = 2.21$ or really $kT = 2.21\,\eta$

```
BoseData =
 {{1, 19/11}, {2, 8/11}, {3, 4/11}, {4, 2/11}, {5, 1/11}, {6, 1/11}, {7, 0}}
```

$$\left\{\left\{1, \frac{19}{11}\right\}, \left\{2, \frac{8}{11}\right\}, \left\{3, \frac{4}{11}\right\}, \left\{4, \frac{2}{11}\right\}, \left\{5, \frac{1}{11}\right\}, \left\{6, \frac{1}{11}\right\}, \{7, 0\}\right\}$$

```
FindFit[BoseData, 1/(Exp[x/T] - 1), {T}, x]
```

$\{T \to 2.21021\}$

see next page for fit with points

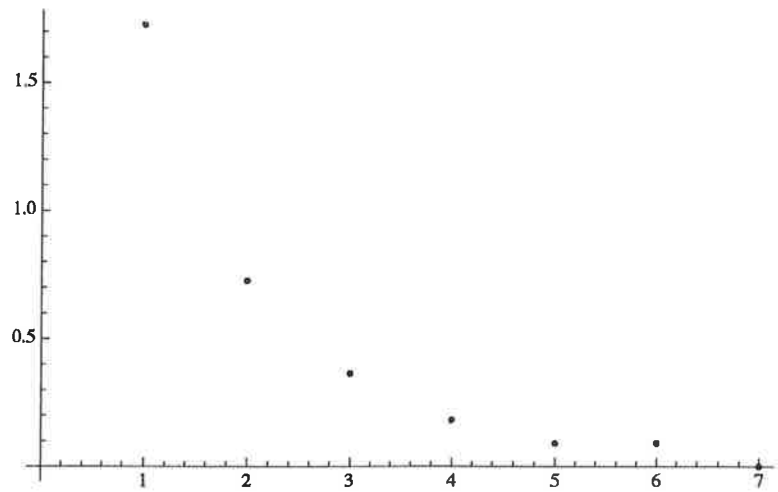
(d) Draw S(E) vs. E + estimate T

Use chart ... and its S(q) vs. q

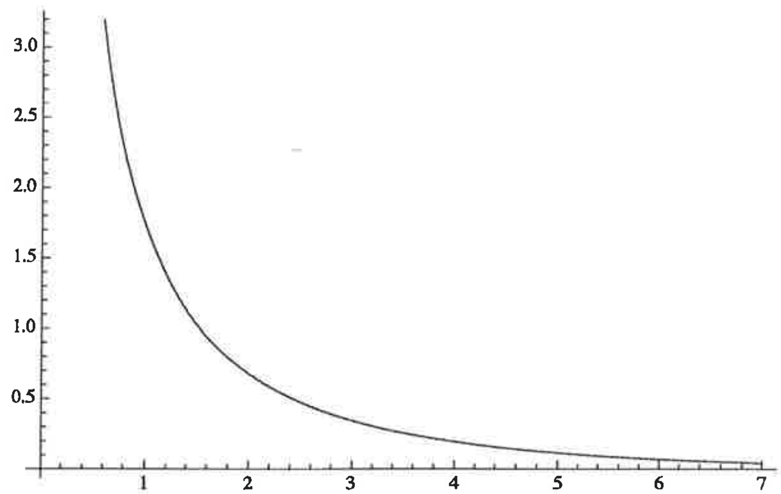
really ... everything scaled by η.

q	Ω	$S/k = \ln \Omega$
0	1	0
1	1	0
2	2	0.69
3	3	1.10
4	5	1.61
5	7	1.95
6	11	2.40

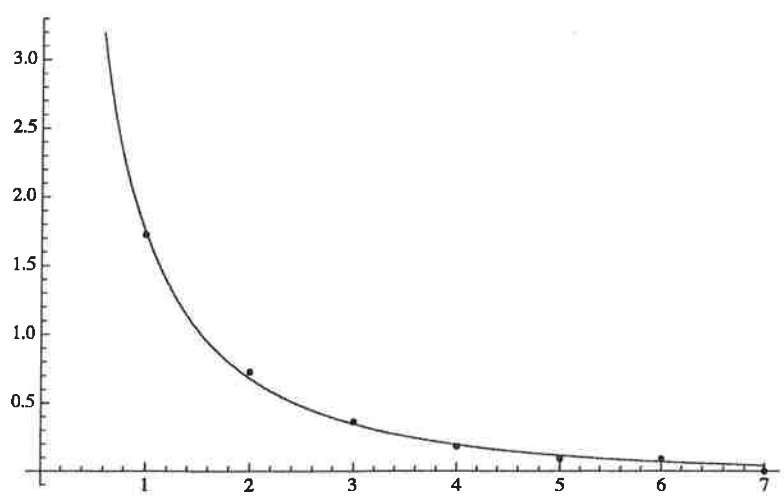
```
plot1 = ListPlot[{19 / 11, 8 / 11, 4 / 11, 2 / 11, 1 / 11, 1 / 11, 0}]
```



```
plot2 = Plot[1 / (Exp[x / 2.21021] - 1), {x, 0, 7}]
```



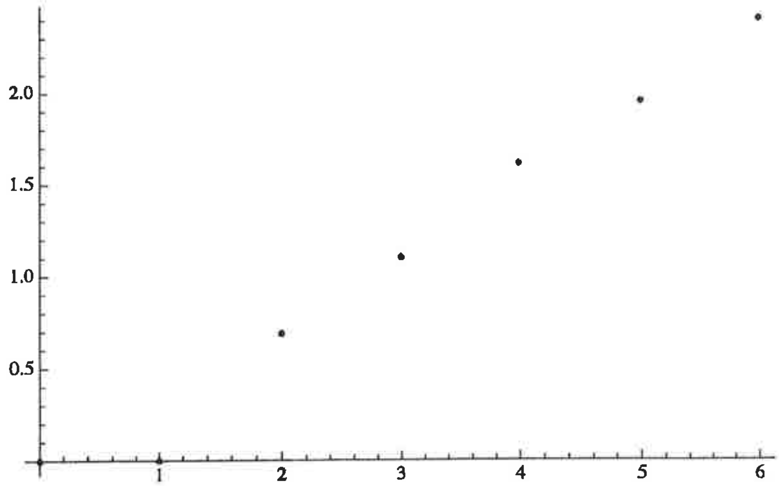
```
Show[plot1, plot2]
```



||

1-5

```
Slist = {{0, 0}, {1, 0}, {2, 0.69}, {3, 1.10}, {4, 1.61}, {5, 1.95}, {6, 2.40}}
{{0, 0}, {1, 0}, {2, 0.69}, {3, 1.1}, {4, 1.61}, {5, 1.95}, {6, 2.4}}
ListPlot[Slist]
```



This graph has slope

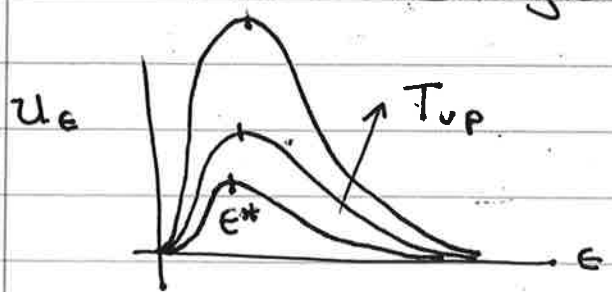
$$\frac{\partial S}{\partial q} = \frac{1}{T} = 0.4293$$

$\Rightarrow T \approx 2.33$ ✓ consistent with fit in part (c)

```
FindFit[Slist, a*x+b, {a, b}, x]
{a → 0.429286, b → -0.180714}
```

Problem 2   B & B 23.3

" Cosmic Microwave
CMB Background "



$$u_\epsilon \sim \frac{\epsilon^3}{e^{\beta\epsilon}-1}$$

In warmup we found peak is at $\epsilon^* \approx 2.82\,kT$

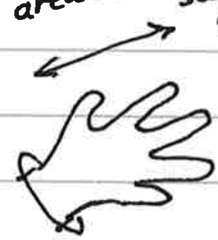
CMB has $T =$ 2.73 K

(a) Energy density

$$\frac{U}{V} = \frac{4\sigma}{c}T^4 = \frac{4 \cdot 5.67\times10^{-8}\ \frac{W}{m^2K^4}\ (2.73\,K)^4}{3\times10^8\ m/s}$$

$$= 4\times10^{-14}\ J/m^3$$

area $\cong 0.01\,m^2$ say



(b) Want the # of photons/sec falling on your hand. Also
(c) Want energy/sec falling.
These problems kind of go together.

$$\frac{\text{Power}}{\text{Area}} = \frac{\text{Energy}}{\text{time}\cdot\text{Area}} = \sigma T^4 = 3.1\times10^{-6}\ \frac{W}{m^2}$$

Answer to (c)

3.1×10^{-8} J/s arrive at palm

Answer to (b)

Energy per photon is on average $(2.7)kT$ (See Table 11 of Left paper, for example.)

so $\frac{U}{N} = 2.7\,kT \cong \underline{1 \times 10^{-22}}\,J$.

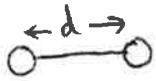
so $\#/\text{area} = \dfrac{3.1 \times 10^{-8}\ J/s}{\underline{1 \times 10^{-22}}\ J} = \dfrac{3 \times 10^{14}\ \frac{\text{photons}}{\text{sec}}}{}$

(d) Radiation Pressure $= \dfrac{1}{3}\dfrac{U}{V} \cong 10^{-14}$ Pa

very small

Problem 3

B & B 29.5



$I = \sum_i m_i r_i^2 = 2 m_p \left(\frac{d}{2}\right)^2 = \frac{1}{2} m_p d^2 = 4.6\times10^{-48}\ \text{kg m}^2 = \mu r^2$ where $r = 2d$

$\mu = \frac{m^2}{m+m} = \frac{m}{2}$ if you prefer!

$k_B \Theta = \frac{2(2+1)\hbar^2}{2I} \Rightarrow \Theta = 524\ \text{K}.$ which is the "rotational temperature"

Triplet state has degeneracy $2\times1+1 = 3$ $\quad J = 1, 3, 5, \ldots$

Singlet " " " $2\times0+1 = 1$ $\quad J = 0, 2, 4, \ldots$

$E_J = \frac{\hbar^2}{2I} J(J+1)$

call this ξ

$\Rightarrow$ Ratio of partition functions

$$f = \frac{3 \sum_{J=1,3,5,\ldots} (2J+1)\, e^{-\beta\xi J(J+1)}}{1 \sum_{J=0,2,4,\ldots} (2J+1)\, e^{-\beta\xi J(J+1)}}$$

$$\approx 3\left[\frac{3e^{-2\beta\xi} + 7e^{-12\beta\xi}}{1 + 5e^{-6\beta\xi}}\right] = 0.27$$

$\beta\xi = 1.75$

(taking first few terms only)

Problem 4 G & T 6.18

Want to find density of states, $g(\epsilon)$ for a 3d relativistic particle of (rest) mass m. For such a particle $\epsilon^2 = p^2c^2 + m^2c^4$

We know that, relativistic or classical, counting states for a particle in a box leads to

$$g(k)\,dk = \frac{V k^2 dk}{2\pi^2}$$ (if $n_s = 1$ ... no additional source of degeneracy)

So want to connect ϵ & k:

Use $p = \frac{h}{\lambda} = \hbar k$ De Broglie

$$\therefore \quad \epsilon^2 = \hbar^2 k^2 c^2 + m^2 c^4$$

$$\Rightarrow \quad k = \frac{1}{\hbar c}\sqrt{\epsilon^2 - m^2c^4}$$

and $$dk = \frac{1}{\hbar c}\cancel{\frac{2}{2}}\epsilon\,(\epsilon^2 - m^2c^4)^{-1/2}\,d\epsilon$$

$$\therefore \quad g(k)\,dk = \underbrace{\frac{V}{2\pi^2}\frac{1}{\hbar^2 c^2}\,\epsilon^2 - m^2c^4\,\frac{1}{\hbar c}\,\epsilon\,(\epsilon^2 - m^2c^4)^{-1/2}}_{\text{all this must be } g(\epsilon)}\,d\epsilon$$

$$\therefore \quad g(\epsilon) = \frac{V\epsilon\sqrt{\epsilon^2 - m^2c^4}}{2\pi^2\hbar^3c^3}$$ for relativistic particle in 3d

Big picture ... compare with non-relativistic particle ...

$$g(\epsilon) \propto \epsilon^{1/2}$$

Also, for relativistic particle with tiny rest mass

$$g(\epsilon) \propto \epsilon^2 \quad \text{(approximately)}$$

We could look back at G + T Section 6.5.1 ... for photons (with $m = 0$), before we take into account the two polariz'n states, we get

$$g(\epsilon)\,d\epsilon = V \frac{\epsilon^2}{2\pi^2\hbar^3 c^3}\,d\epsilon \qquad (6.99)$$

perfect!! :)

G & T 6.20

As done in Prob 6.19 for non-relativistic, massive particles, want to show that for photons:

$$PV = \frac{1}{3}\bar{E}$$

(vs. $PV = \frac{2}{3}E$ for former)

$$\bar{E} = \int_0^\infty \epsilon\,\bar{n}(\epsilon)\,g(\epsilon)\,d\epsilon$$

$$= n_s \frac{V}{2\pi^2\hbar^3 c^3}\int_0^\infty \frac{\epsilon^3\,d\epsilon}{e^{\beta\epsilon}-1}$$

where $g(\epsilon)$ is as above, (6.99)

and $n_s = 2$ ...

and $\bar{n}(\epsilon)$ is the Boson occupation number function, with $\mu \equiv 0$. (Photons can be created + destroyed at will.)

Now, for photons

$$\Omega = +kT\int_0^\infty g(\epsilon)\ln\left[1-e^{-\beta\epsilon}\right]d\epsilon$$

$$= \frac{kTn_sV}{2\pi^2\hbar^3c^3}\int_0^\infty \epsilon^2\ln\left[1-e^{-\beta\epsilon}\right]d\epsilon$$

Leave the constant prefactor aside for a moment ... we have a form $C\int_0^\infty u\,dv = \Omega$

where $u = \ln\left[1-e^{-\beta\epsilon}\right]$

$dv = \epsilon^2 d\epsilon$

$\Rightarrow v = \frac{1}{3}\epsilon^3$ and $du = \dfrac{\beta e^{-\beta\epsilon}}{1-e^{-\beta\epsilon}}d\epsilon = \dfrac{\frac{1}{kT}}{e^{\beta\epsilon}-1}d\epsilon$

$$\therefore \Omega = C\int_0^\infty u\,dv = C\,uv\Big|_0^\infty - C\int_0^\infty v\,du$$

Thus $$\Omega = \frac{kTn_sV}{2\pi^2\hbar^3c^3}\left[\ln\left[1-e^{-\beta\epsilon}\right]\frac{\epsilon^3}{3}\Big|_0^\infty - \frac{1}{3}\int_0^\infty\frac{\epsilon^3/kT}{e^{\beta\epsilon}-1}d\epsilon\right]$$

Lower limit ... $\epsilon \to 0$ ... This expression becomes $\ln\left[1-(1-\beta\epsilon+\cdots)\right]\dfrac{\epsilon^3}{3}$

$$= \ln(\beta\epsilon)\frac{\epsilon^3}{3}$$

Now $\ln \to -\infty$ but much more slowly than $\epsilon \to 0$. So lower limit vanishes.

Upper limit ... $\epsilon \to \infty$ ... This expression becomes $\ln\left[1-\eta\right]\left(\dfrac{-kT\ln\eta}{3}\right)^3 \approx \dfrac{(kT)^3}{3}\eta(\ln\eta)^3$

Now $\ln \to -\infty$ but much more slowly than η itself $\to 0$

$-\beta\epsilon = \eta$

$-\beta\epsilon = \ln\eta$

$\epsilon = -kT\ln\eta$

where $\eta \to 0$

Carrying on,

$$\Omega = -\frac{1}{3}\frac{\cancel{kT}\, n_s V}{2\pi^2\hbar^3 c^3\, k\cancel{T}}\int_0^\infty \frac{\epsilon^3 d\epsilon}{e^{\beta\epsilon}-1}$$

$$= -\frac{1}{3}\bar{E}$$

from p. 4-2.

Since $\Omega = -PV$ we thus have:

$$PV = \frac{1}{3}\bar{E} \quad \text{QED}$$

Problem 5 Schroeder 7.44

Number of photons in gas

(a) Want to show

$$N = 8\pi V \left(\frac{kT}{hc}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1}\, dx .$$

This is done in G & T reading ... section 6.7

There, they observe that the number of photons within $d\epsilon$ of energy ϵ is

$$N(\epsilon)\, d\epsilon = \bar{n}(\epsilon)\, g(\epsilon)\, d\epsilon$$

$$= \frac{1}{e^{\beta\epsilon} - 1} \cdot \frac{n_s V}{2\pi^2 \hbar^3 c^3} \epsilon^2 d\epsilon$$

as in section 6.5.1 + used in $m=0$ limit of problem 6.18

where we will use $n_s = 1$. Thus

$$N = \int_0^\infty N(\epsilon)\, d\epsilon = \frac{V}{\pi^2 \hbar^3 c^3} \int_0^\infty \frac{\epsilon^2}{e^{\beta\epsilon} - 1}\, d\epsilon$$

Let us change variable ...

$$x = \beta\epsilon \Rightarrow \frac{dx}{\beta} = d\epsilon \quad ; \quad \frac{x}{\beta} = \epsilon$$

$$\therefore \quad N = \frac{V}{\pi^2 \hbar^3 c^3} \frac{1}{\beta^3} \int_0^\infty \frac{x^2}{e^x - 1}\, dx$$

$$N = \frac{8\pi^3 (kT)^3 V}{\pi^2 h^3 c^3} \int_0^\infty \frac{x^2}{e^x - 1}\, dx = 8\pi V \left(\frac{kT}{hc}\right)^3 \underbrace{\int_0^\infty \frac{x^2\, dx}{e^x - 1}}_{\Gamma(3)\zeta(3)}$$

QED

This is from Schroeder App B Eq. B.36 $\int_0^\infty \frac{x^n}{e^x - 1} dx = \Gamma(n+1)\zeta(n+1)$. Unfortunately the ζ function can't be analytically evaluated for n odd ... so we just go with a numerical evaluation. This integral is 2.404

(b) We want S/N.
Schroeder finds entropy, S, by starting with $\bar{E}$ as in Eq. (7.85) (and many other places this week ☺)

$$\bar{E} = \frac{8\pi (kT)^4}{(hc)^3} \underbrace{\int_0^\infty \frac{x^3}{e^x - 1}\, dx}_{\Gamma(4)\zeta(4) = \frac{\pi^4}{15}}$$

Now, $S(T) = \int_0^T \frac{C_V(T')}{T'}\, dT'$ and

$$C_V(T') = \left(\frac{\partial E}{\partial T'}\right)_V$$

Putting these together:

$$S(T) = \frac{32\pi^5}{45} V \left(\frac{kT}{hc}\right)^3 k$$

Thus $\frac{S(T)}{N(T)} = \frac{32\pi^5/45}{8\pi\, 2.404}\, k = \underline{3.602\, k}$

This is average entropy per photon

(c) At 300K,

$$\frac{N}{V} = 2.404 \cdot 8\pi \left(\frac{kT}{hc}\right)^3$$

$$= \underline{5.5 \times 10^{14} \frac{\text{photons}}{m^3}}$$

Seems big, but is small cf. density of ideal gas at $P = 1$ atm, $T = 300K$.
Since $\frac{N}{V} \propto T^3$, at $T = 1500$ K, $\cancel{\frac{N}{V} = \left(\frac{N}{V}\right)_{T=300K}}$

$$\frac{N}{V} = (5^3)(5.5 \times 10^{14}) \frac{\text{photons}}{m^3}$$

$$= 6.8 \times 10^{16} \frac{\text{photons}}{m^3}$$

At $T = 2.73$K, $\frac{N}{V} = \left(\frac{2.73}{300}\right)^3 (5.5 \times 10^{14}) = \underline{4.1 \times 10^8 \frac{\text{photons}}{m^3}}$

This is large, cf. matter density of universe!

Schroeder 7.46

Finding the free energy (Helmholtz) of photon gas:

(a) From equations 7.86 and 7.89, we have

$$F = U - TS = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}V - T\cdot\frac{32\pi^5}{45}V\left(\frac{kT}{hc}\right)^3 k$$
$$= \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}V\left(1-\frac{4}{3}\right) = -\frac{8\pi^5}{45}\frac{(kT)^4}{(hc)^3}V = -\frac{1}{3}U.$$

(b) Differentiating this result with respect to T gives

$$\left(\frac{\partial F}{\partial T}\right)_V = -\frac{32\pi^5}{45}\frac{k^4T^3}{(hc)^3}V,$$

which is indeed equal to $-S$, by equation 7.89.

(c) By equation 5.22 and the result of part (a),

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{8\pi^5}{45}\frac{(kT)^4}{(hc)^3} = \frac{1}{3}\frac{U}{V},$$

in agreement with the result of the previous problem.

(d) For any particular mode with energy ϵ, the partition function is $Z = (1-e^{-\epsilon/kT})^{-1}$, as calculated in equation 7.70. Therefore the free energy of this mode is $F = -kT\ln Z = kT\ln(1-e^{-\epsilon/kT})$. To get the total free energy, we sum this expression over all modes, as in equation 7.81:

$$F = 2\sum_{n_x,n_y,n_z} kT\ln(1-e^{-\epsilon/kT}) = 2kT\cdot\frac{\pi}{2}\int_0^\infty n^2\ln(1-e^{-\epsilon/kT})\,dn.$$

In the last expression I've converted the sum to an integral in spherical coordinates over the first octant of n-space, and carried out the angular integrals to obtain $\pi/2$, the area of an eighth of a unit sphere. Changing variables to $x = \epsilon/kT = hcn/2LkT$ then gives

$$F = \pi kT\left(\frac{2LkT}{hc}\right)^3\int_0^\infty x^2\ln(1-e^{-x})\,dx = 8\pi V\frac{(kT)^4}{(hc)^3}\int_0^\infty x^2\ln(1-e^{-x})\,dx.$$

To put this integral into a more familiar form, integrate by parts; that is, integrate the x^2 to obtain $x^3/3$, and differentiate the logarithm:

$$F = 8\pi V\frac{(kT)^4}{(hc)^3}\left[\frac{x^3}{3}\ln(1-e^{-x})\Big|_0^\infty - \int_0^\infty \frac{x^3}{3}\frac{e^{-x}}{1-e^{-x}}\,dx\right]$$

The boundary term vanishes at both limits, so we're left with

$$F = -\frac{8\pi V}{3}\int_0^\infty \frac{x^3}{e^x-1}\,dx = -\frac{1}{3}U,$$

by comparison with equation 7.85. This is the same result obtained in part (a).

Schroeder 7.49

Problem 6 → e^+ & e^- in early universe

(a) WTS $\frac{U}{V} = 16\pi \frac{(kT)^4}{(hc)^3} u(T)$; $u(T) = \int_0^\infty \frac{x^2 \sqrt{x^2 + (mc^2/kT)^2}}{e^{\sqrt{x^2+(mc^2/kT)^2}} + 1} dx$

We did an earlier problem, G & T 6.18 where we found $g(\epsilon)$ so will not reinvent the wheel. We'll use that form & assert

$$U = \bar{E} = \underset{\substack{\uparrow \\ \text{positrons}}}{U_+} + \underset{\substack{\uparrow \\ \text{electrons}}}{U_-} = \int_{\epsilon_{min}}^{\infty} \epsilon\, \bar{n}_+(\epsilon)\, g(\epsilon)\, d\epsilon + \int_{\epsilon_{min}}^{\infty} \epsilon\, \bar{n}_-(\epsilon)\, g(\epsilon)\, d\epsilon$$

Note... in relativity $\epsilon_{min} \neq 0$; but $\epsilon_{min} = mc^2$

Now, we will argue that $g(\epsilon)$, as is written above, is same for e^+ & e^- since $m_{e+} = m_{e-}$ and g is just about counting possible states in volume V.

Also, since $e^+ + e^- \rightarrow 2\gamma$

$$\mu_{e+} + \mu_{e-} = 2\mu_\gamma = 0$$

And by symmetry, $\mu_{e+} = \mu_{e-}$. Thus, we have same chemical pot'l for e^+ & e^- and it is zero. Thus

$$\bar{n}_+(\epsilon) = \bar{n}_-(\epsilon) = \frac{1}{e^{\epsilon/kT}+1}$$ fermions!

Now, we will have a degeneracy of $n_s = 2$ for these species due to spin of $1/2$. So beginning with relevant $g(\epsilon)$ from G+T 6.18 and summing U_+ and U_- (so get an additional factor of 2) we have

$$U = \frac{4V}{2\pi^2(\hbar c)^3} \int_{mc^2}^{\infty} \frac{\epsilon^2 \sqrt{\epsilon^2-(mc^2)^2}\, d\epsilon}{e^{\epsilon/kT}+1} \quad (1)$$

To make Eq. (1) look like starting form, two steps are needed:

#1 change from $\hbar$ to h in denom:

$$(1) \Rightarrow \frac{U}{V} = \frac{16\pi}{(hc)^3} \int_{mc^2}^{\infty} \frac{\epsilon^2 \sqrt{\epsilon^2-(mc^2)^2}\, d\epsilon}{e^{\epsilon/kT}+1} \quad (2)$$

#2 change variable. Clearly the $e^{\epsilon/kT}$ should be $e^{\sqrt{x^2+\left(\frac{mc^2}{kT}\right)^2}}$ so

$$\epsilon = kT\sqrt{x^2+\left(\frac{mc^2}{kT}\right)^2}$$

$$\Rightarrow \quad x = \frac{1}{kT}\sqrt{\epsilon^2-(mc^2)^2}$$

$$\Rightarrow dx = \frac{\epsilon/kT \quad d\epsilon}{\sqrt{\epsilon^2-(mc^2)^2}} \quad ; \quad d\epsilon = \frac{kT\,x\,kT\;dx}{kT\sqrt{x^2+\left(\frac{mc^2}{kT}\right)^2}}$$

$$\therefore \frac{U}{V} = \frac{16\pi}{(hc)^3}\int_0^\infty \frac{(kT)^2\left(x^2+\left(\frac{mc^2}{kT}\right)^2\right)}{e^{\sqrt{x^2+\left(\frac{mc^2}{kT}\right)^2}}+1}\,\frac{kTx\;kTx\,dx}{\sqrt{x^2+\left(\frac{mc^2}{kT}\right)^2}},$$

or

$$\frac{U}{V} = \frac{16\pi(kT)^4}{(hc)^3}\underbrace{\int_0^\infty \frac{x^2\sqrt{x^2+\left(\frac{mc^2}{kT}\right)^2}\,dx}{e^{\sqrt{x^2+\left(\frac{mc^2}{kT}\right)^2}}+1}}_{u(T)}$$

QED! :)

(b) Rewrite $u(T)$ using $t = \frac{kT}{mc^2}$

$$u(t) = \int_0^\infty \frac{x^2\sqrt{x^2+(1/t)^2}}{e^{\sqrt{x^2+(1/t)^2}}+1}\,dx$$

When $kT \gg mc^2$, $t \ll 1$. In this limit, $e^{\sqrt{x^2+(1/t)^2}}$ is large (over all values of $x \in (0,\infty)$). For this reason, $u(t) \to 0$ exponentially as $e^{-1/t}$ with $t \to 0$. This makes sense b/c need at least rest energy to create the pair.

(c) As $t\to\infty$, can neglect terms in $1/t$ to get

$$u(t) \to \int_0^\infty \frac{x^3}{e^x+1}\,dx = \left(1-\frac{1}{2^3}\right)\Gamma(4)\zeta(4) = \frac{7}{8}\frac{\pi^4}{15} \approx 5.68$$

While we didn't do previous problem on neutrinos, we can compare with Planck photon spectrum...

$$\frac{U}{V} = \frac{8\pi(kT)^4}{(hc)^3}\int_0^\infty \frac{x^3}{e^x-1}\,dx = \frac{8\pi(kT)^4}{(hc)^3}\Gamma(4)\zeta(4) \quad \text{Planck Photons}$$

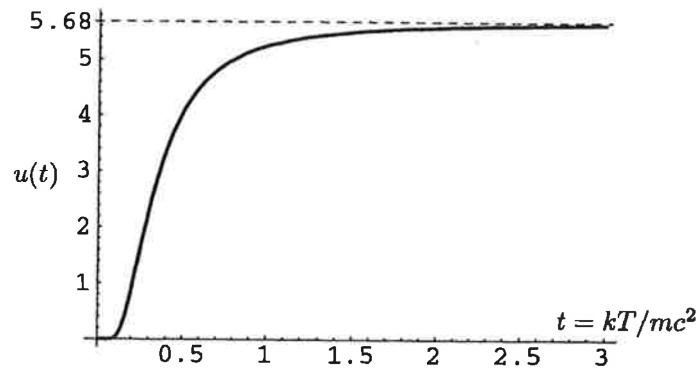
vs e^+/e^- as $T\to\infty$ $\quad \frac{U}{V} = \frac{16\pi(kT)^4}{(hc)^3}\frac{7}{8}\Gamma(4)\zeta(4)$ $\quad$ Cool!

So $U/V \approx \dfrac{14\pi^5(kT)^4}{15(hc)^3}$

(d) To plot $u(T)$ I used the *Mathematica* instructions

```
u[t_] := NIntegrate[
    x^2*Sqrt[x^2+t^-2]/(Exp[Sqrt[x^2+t^-2]]+1),{x,0,Infinity}]
Plot[u[t],{t,0,3}]
```

which produced the following plot:



(The `Plot` instruction generated several error messages, because the exponential function overflows at small t values. Nevertheless, the plotted curve correctly shows that $u(t)$ is essentially zero below $t = 0.1$.) I added the dashed line to the plot, to show the asymptotic value calculated in part (c).

(e) As in Problem 7.46(d), consider first just a single "mode" (or single-particle state), which can be occupied either by zero particles (with energy zero) or one particle (with energy ϵ). The partition function of this mode is $Z = 1 + e^{-\epsilon/kT}$, and the free energy is $F = -kT\ln Z = -kT\ln(1 + e^{-\epsilon/kT})$. To obtain the *total* free energy, we sum this expression over all modes:

$$F = 4\sum_{n_x,n_y,n_z}(-kT)\ln(1+e^{-\epsilon/kT}) = -4kT\cdot\frac{\pi}{2}\int_0^\infty n^2\ln(1+e^{-\epsilon/kT})\,dn$$

$$= -2\pi(kT)\left(\frac{2LkT}{hc}\right)^3\int_0^\infty x^2\ln(1+e^{-\epsilon/kT})\,dx = -\frac{16\pi(kT)^4}{(hc)^3}f(T),$$

where

$$f(T) = \int_0^\infty x^2\ln(1+e^{-\epsilon/kT})\,dx = \int_0^\infty x^2\ln\left(1+e^{-\sqrt{x^2+(1/t)^2}}\right)dx.$$

As $T \to 0$, the exponential factor becomes very small for all x, so we can expand the logarithm: $\ln(1+e^{-\epsilon/kT}) \approx e^{-\epsilon/kT}$. This exponential factor therefore suppresses the entire expression for $f(T)$, so the free energy, like the energy, vanishes when the temperature is too low to create electron-positron pairs. In the other limit, where $t \gg 1$, we can neglect the $1/t$ term in the exponent to obtain simply

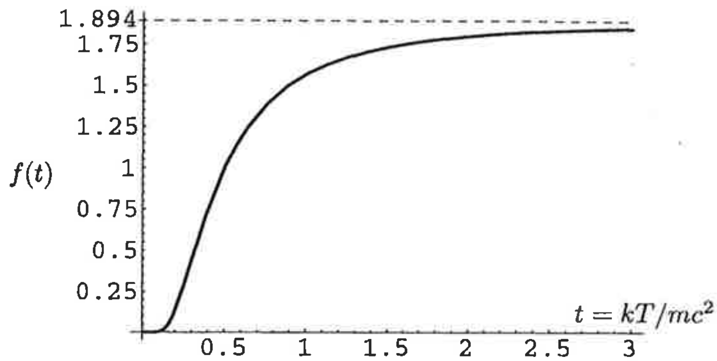
$$f(T) \to \int_0^\infty x^2\ln(1+e^{-x})\,dx = -\int_0^\infty \frac{x^3}{3}\frac{-e^{-x}}{1+e^{-x}}\,dx$$

$$= \frac{1}{3}\int_0^\infty \frac{x^3}{e^x+1}\,dx = \frac{1}{3}\cdot\frac{7}{8}\cdot\frac{\pi^4}{15} = \frac{7\pi^4}{360} \approx 1.894.$$

(In the second step I've integrated by parts and dropped the boundary term which vanishes at both limits.) To plot $f(T)$ I used the *Mathematica* instructions

```
f[t_] := NIntegrate[x^2*Log[1+Exp[-Sqrt[x^2+t^-2]]],{x,0,Infinity}]
Plot[f[t],{t,0,3}]
```

which (after a long list of nonfatal error messages) produced the following:



(f) From the definition $F = U - TS$, we have simply

$$S = \frac{U-F}{T} = \frac{1}{T}\left(\frac{16\pi V(kT)^4}{(hc)^3}u(T) + \frac{16\pi V(kT)^4}{(hc)^3}f(T)\right) = \frac{16\pi V(kT)^3}{(hc)^3}\big(u(T)+f(T)\big)k.$$

When $T \ll mc^2$, this expression goes exponentially to zero along with $u(T)$ and $f(T)$. In the high-temperature limit, it goes to

$$S \to 16\pi V\left(\frac{kT}{hc}\right)^3 \cdot \frac{7}{8}\cdot\frac{\pi^4}{15}\left(1+\frac{1}{3}\right)k = \frac{56\pi^5}{45}V\left(\frac{kT}{hc}\right)^3 k.$$