Phys 114 Seminar 4 sol'ns

Problem 1

I ♡ Random numbers

Intro...

Question 1. (Emma) Uniform probability distribution

1. For each of 500 sets of 200 randomly generated numbers, find the sum. Plot the distribution of the sums.

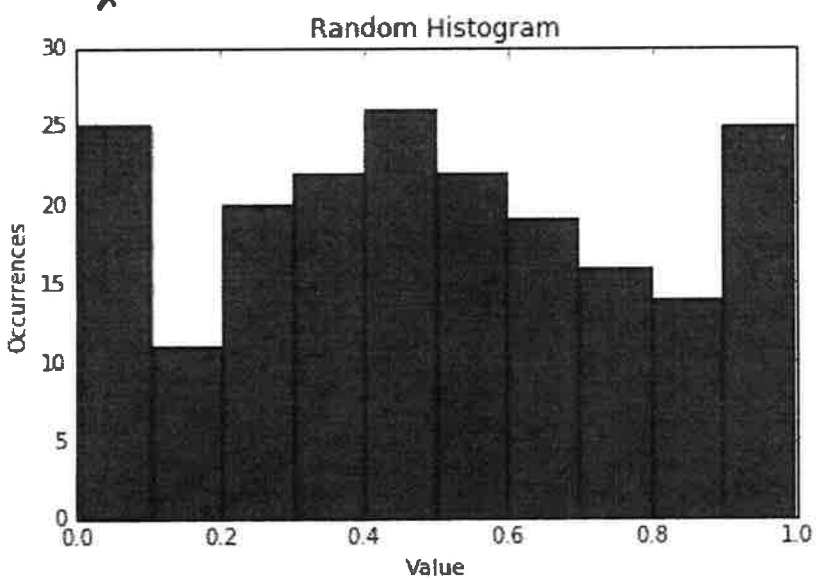
Is the distribution Gaussian? Does the distribution have the mean and variance in agreement with the central limit theorem?

- Mean of the sum distribution should be N*(mean of original distribution)
- Variance = N*(original variance) as variance is additive

$$\bar{X} = \sum_{n=1}^{N=200} x_n$$

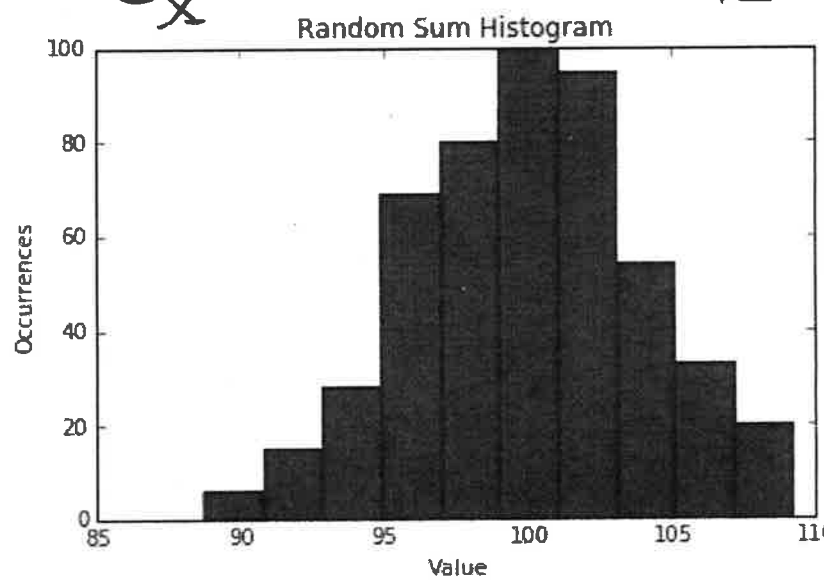
$\bar{X}$: Mean is close to $1/2$

σ_x^2 : Variance close to $1/2$



$\bar{X}$ Mean close to $200 \times 1/2$

$\sigma_{\bar{X}}^2$: close to $200 \times 1/12$



Problem 2

RW's and the binomial dist'n

a) G&T 3.35

RWer takes n steps R
n' steps L so $n+n'=N$

Step has length a. Prob step to right is p. What is $\bar{x}$? $\overline{\Delta x^2}$?

Answer: n + n' are a bit of a smokescreen... anyway

$$x = a(n-n') = a(2n-N)$$

So find $\bar{x} = a(2\bar{n} - N)$.

For Bernoulli process, $P_N(n) = \binom{N}{n} p^n (1-p)^{N-n}$

This is written about at length in G&T & B&B. The mean is $Np = \bar{n}$

Thus $\bar{x} = a(2Np - N) = aN(2p-1)$

reality check: $p = 1/2 \Rightarrow \bar{x} = 0$

Similarly, $x^2 = a^2(2n-N)^2$

$$\overline{x^2} = a^2(4\overline{n^2} - 2N\bar{n} + N^2)$$

and $\overline{n^2} = \bar{n}^2 + p(1-p)N$ G&T p. 135 Binomial dist

$$\Rightarrow \overline{x^2} = a^2[4\bar{n}^2 + 4p(1-p)N - 2N\bar{n} + N^2]$$

$$\Rightarrow \overline{x^2} - \bar{x}^2 = \overline{\Delta x^2} = a^2[4p(1-p)N]$$ Thus $\overline{\Delta x^2} \propto N$ and $\bar{x} \propto N$ too!

Problem 3

i) What are the rules?

$$\sum_i P_i = 1$$

$$P_{\neg i} = 1 - P_i$$

$$P_{i \text{ or } j} = P_i + P_j$$

$$P_{i \text{ and } j} = P_i P_j$$

$$\bar{A} = \sum_i P_i A_i$$

ii) G + T Prob 3.18

	yes	no	no	
	yes	yes	yes	No
Two children ... would be	GG	GB	BG	BB

Write out microstates

(a) Prob at least one is a girl? $\underline{3/4}$

(b) Know at least one is a girl. Prob other is also a girl? $\underline{1/3}$

(c) Know Oldest child is a girl. Prob youngest is a girl?

	yes		no	
youngest/oldest:	GG	GB	BG	BB

$\underline{1/2}$

Problem 3 (cont) (iii) B+B Prob 3.5 (a), (c), (e)

Θ is cts RV uniform on $(0, \pi)$

(a) $$\langle \theta \rangle = \frac{1}{\pi}\int_0^\pi \theta \, d\theta = \frac{1}{\pi}\frac{\theta^2}{2}\Big|_0^\pi$$

normaliz'n

$$= \frac{\pi}{2}$$

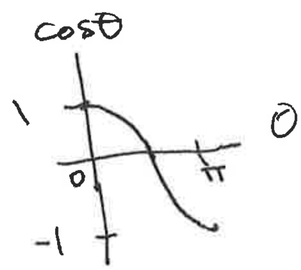
(c) $$\langle \theta^2 \rangle = \frac{1}{\pi}\int_0^\pi \theta^2 \, d\theta = \frac{1}{\pi}\frac{\theta^3}{3}\Big|_0^\pi$$

$$= \frac{\pi^2}{3}$$

(e) $$\langle \cos\theta \rangle = \frac{1}{\pi}\int_0^\pi \cos\theta \, d\theta$$

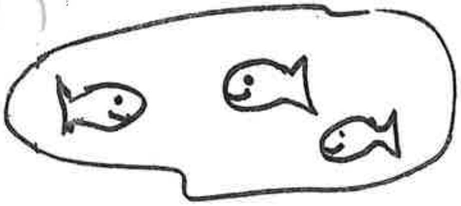
$$= \frac{1}{\pi}\sin\theta\Big|_0^\pi = 0$$

Makes sense



Problem 4)

i) G+T 3.58



200 fish tagged ... put back.
250 taken out ... 25 tagged.

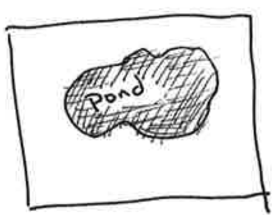
$\frac{25}{250} = \frac{1}{10}$. So 200 is $\frac{1}{10}$ of # fish. or $P_{tag} = \frac{1}{10}$

$$\bar{n}_{tag} = N P_{tag} = 200 = \frac{N}{10}$$

$\Rightarrow$ $\underline{N = 2000}$

ii) G+T 3.59

10 m

field  10 m

Area of pond?
Throw 100 stones and 40 make a splash.

$$\frac{40}{100} = \frac{\text{area of pond}}{\text{area of field}} = \frac{\text{Area of pond}}{100\ m^2}$$

$\Rightarrow$ Pond is $\underline{40\ m^2}$ in area

Presentation Problem 5) (P)

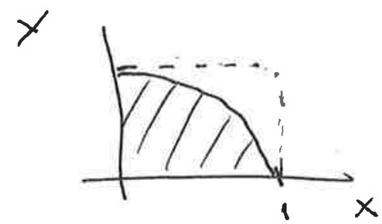
G&T 3.60 Monte Carlo Integration

TABLE 3.6
A sequence of ten random pairs of numbers (see Problem 3.60).

	x_i, y_i		x_i, y_i
1	0.984, 0.246	6	0.637, 0.581
2	0.860, 0.132	7	0.779, 0.218
3	0.316, 0.028	8	0.276, 0.238
4	0.523, 0.542	9	0.081, 0.484
5	0.349, 0.623	10	0.289, 0.032

To integrate $F = \int_0^1 dx \sqrt{1-x^2}$

This is area under semicircle



To we approx with $\frac{\text{\# dots inside}}{\text{total \# dots}}$

looking at list above, which of y values are st $y^2 < 1 - x^2$?

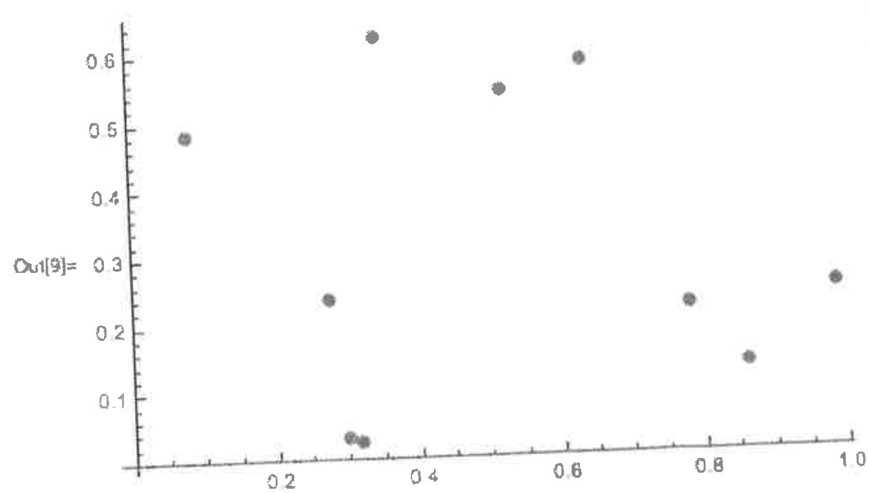
1 NO
2 YES
3 YES
4 YES
5 YES
6 YES
7 YES
8 YES
9 YES
10 YES

$\Rightarrow F \simeq \frac{9}{10} = 0.9$

Right answer: $\frac{\pi}{4} = 0.79$

This kind of sucks... "random" points have too low Y Values!

```
In[8]:= data = Transpose@{x, y};
ListPlot[data]
```



— $y_{max} = 0.623$

(of course, if only 1 point were moved down, we'd have $F \simeq 0.8$ ◡̈

Prob .. (cont)

Lets get to other tasks ...

(a) WTS : $F = \frac{\pi}{4}$

Δ of variable : $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$\Rightarrow F = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \underbrace{\cos^2\theta}_{\text{has ave of } \frac{1}{2}}\, d\theta = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4} = 0.785\ldots$$

QED

(b) Use MonteCarlo Estimation. Find error for trials with $n = 10^4, 10^6, 10^8$. How does error scale with n?

n	$F_{estimated}$ *	$\left\lvert\left(\overline{F}_{estimated} - \frac{\pi}{4}\right)\right\rvert$
1000	0.773, 0.791	.004
10^4	0.7819, 0.7845	.002
10^5	0.78302, 0.78579	.001
10^6	0.785578, 0.785126	.0005

* Note Different trials yield different values! Just do two + take $\overline{F}_{estimated}$

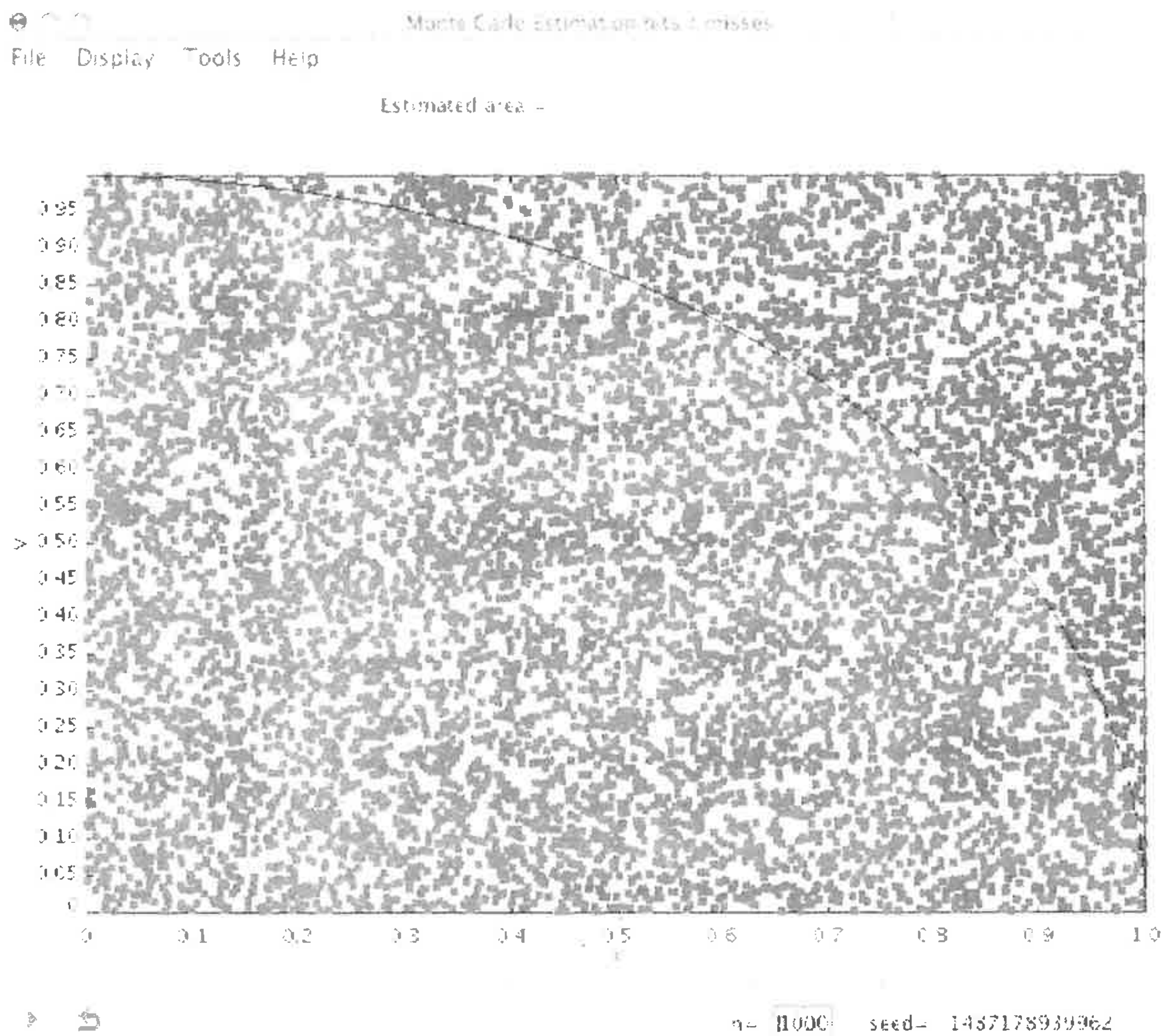
doesn't look like I can go higher with this code :(

yes, error does decrease with n ... on average!

Pres Prob (cont'd)

MC integration applic'n

ugh, I should have given window a square aspect ratio!



$n = 10^4$ here

Prob - (cont)

(c) Estimate the integral using $n = 1000$. Repeat for a total of 10 trials using different random number seeds each time. The easiest way to do so is to press the Reset button and then press the Calculate button. The default is for the program to choose a new seed each time based on the time. Is the magnitude of the variation of your values of the same order as the error between the average value and the exact value? For a large number of trials, the error is estimated from the standard error of the mean, which approximately equals the standard deviation divided by the square root of the number of trials.

Solution. The average of the data in Table S3.9 is 0.7837 compared to the exact value $\pi/4 \approx 0.7854$. The difference between these two is $0.0017 \approx 0.002$. The data are about 0.01 from the average, which would predict an error of about $0.01/\sqrt{10} \sim 0.003$, which is consistent with the deviation of the average from the exact value.

meaning $\sigma \simeq 0.01$

Trial	Estimate
1	0.794
2	0.757
3	0.793
4	0.780
5	0.796
6	0.795
7	0.775
8	0.772
9	0.798
10	0.777

Table S3.9: The results of estimating the integral in (3.194) for $n = 1000$ and ten different random number seeds. As expected, the estimate is slightly different each time the program is run.

Problem 6

(a) Poisson dist'n

$$P(x) = \frac{e^{-m} m^{x}}{x!}$$

(a) TS $\sum_{x=0}^{\infty} P(x) = 1$

This summation is

$$e^{-m} \underbrace{\sum_{x=0}^{\infty} \frac{m^x}{x!}}_{\text{Taylor expan of } e^m} = e^{-m} e^{m} = 1 \checkmark$$

(b) TS $\langle x \rangle = m$.

$$\langle x \rangle = \sum_x x P(x) = e^{-m} \sum_{x=0}^{\infty} \frac{x\, m^x}{x!}$$

since $x=0$ term does not contribute

$$= e^{-m} \sum_{x=1}^{\infty} \frac{x\, m^x}{x!}$$

let $y = x-1$

$$= e^{-m} \sum_{x=1}^{\infty} \frac{m\, m^{x-1}}{(x-1)!} = e^{-m} \sum_{y=0}^{\infty} \frac{m\, m^{y}}{y!}$$

$$= m \cdot 1 = m \checkmark$$

(c) Horse kicks in prussian army: 10 corps-worth of data added 1875 - 1894

I don't believe this... it could be a fraction divide by 10

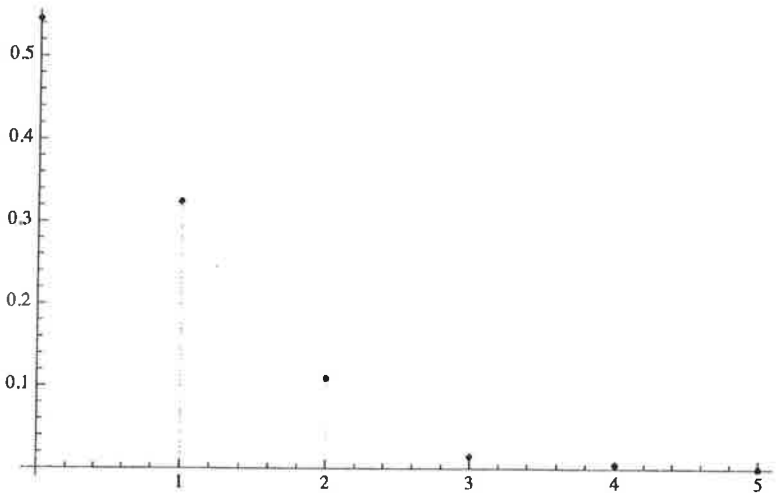
# deaths per year, per corps	frequ
0	109
1	65
2	22
3	3
4	1
≥ 5	0
total	200

Now Poisson distribution ... Problem B+B 3.3

```
deathlist =
 {{0, 109 / 200}, {1, 65 / 200}, {2, 22 / 200}, {3, 3 / 200}, {4, 1 / 200}, {5, 0}}
```

$$\left\{\left\{0, \frac{109}{200}\right\}, \left\{1, \frac{13}{40}\right\}, \left\{2, \frac{11}{100}\right\}, \left\{3, \frac{3}{200}\right\}, \left\{4, \frac{1}{200}\right\}, \{5, 0\}\right\}$$

```
a = ListPlot[deathlist, Filling → Axis]
```




```
1 * 65 + 2 * 22 + 3 * 3 + 4 * 1
```

122

```
% / 4
```

```
122 / 200
```

$\frac{61}{100}$

```
N[%]
```

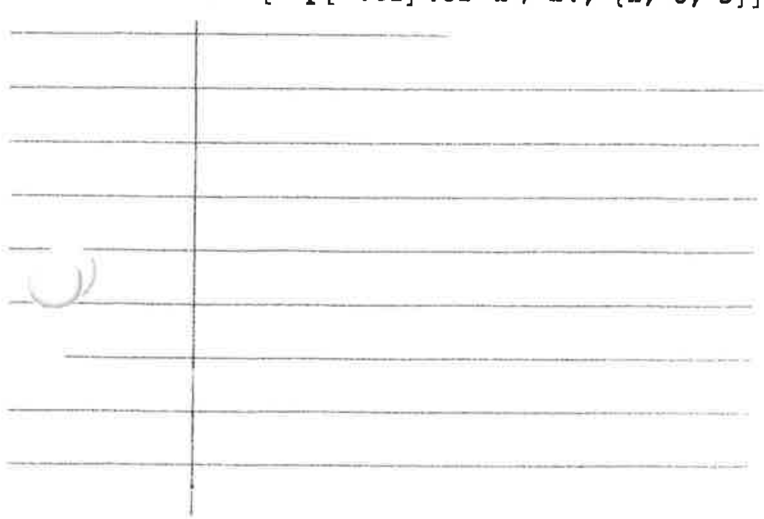
0.61

This is the mean number of deaths per year ... 0.61 soldier

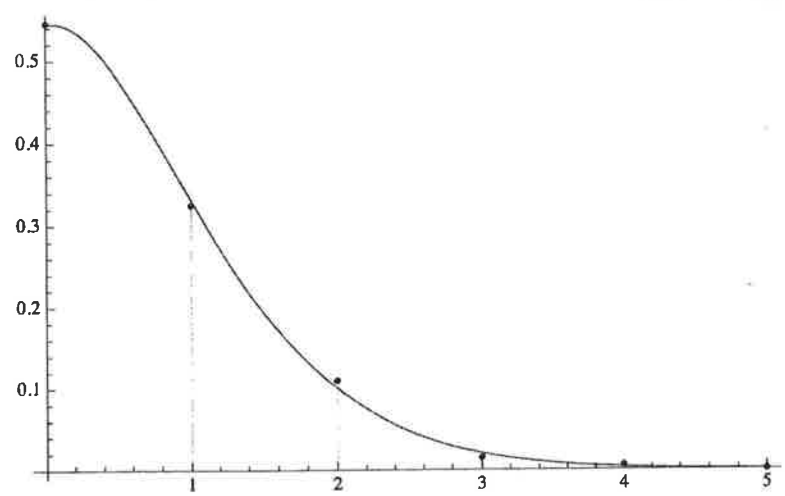
Here is the poisson distribution with that mean

```
Poiss[x_] := Exp[-.61] .61^x / x!
```

```
b = Plot[Exp[-.61] .61^x / x!, {x, 0, 5}]
```



```
Show[a, b]
```



Both model distn & numbers ... not bad!

Problem 7

i) Binomial dist'n for Magnets & Gasses

G&T 3.27 Indep magnetic moments

& N = 3 ↑↓↑ book showed

$\overline{\Delta M^2} = 3\overline{(\Delta S)^2}$ b/c cross terms

like $\overline{\Delta S_1 \Delta S_2} = \overline{\Delta S_1}\,\overline{\Delta S_2}$

BTW $= 3 \cdot 4pq$ $= 0$ ... cancel

$(\Delta S_i = S_i - \overline{S_i})$

To show: This generalizes

for to $\overline{\Delta M^2} = N \cdot 4pq$

proof $\Delta M^2 = \sum_i \Delta S_i \sum_j \Delta S_j$

$$= \sum\sum_{i \neq j} \Delta S_i \Delta S_j + \sum_{i=j} \Delta S_i^2$$

So $$\overline{\Delta M^2} = \sum\sum_{i \neq j} \overline{\Delta S_i \Delta S_j} + \sum_{i=j} \overline{\Delta S_i^2}$$

independence of spin

$\overline{\Delta S_i}\,\overline{\Delta S_j}$ $\overset{?}{=}$

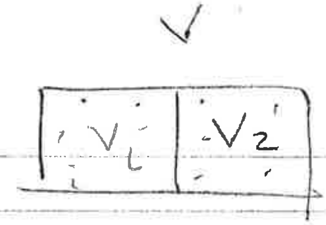
$\parallel$

0

Thus $\overline{\Delta M^2} = N\overline{\Delta S_i^2} = N \cdot 4pq$

Prob 7
ii)

G&T 3.34



gas ...

(a) Prob in V_1 ?

For any particular molecule it is V_1/V.

(b) Prob N_1 in V_1 + $N_2 = N - N_1$ in V_2 ?

If particles are indep, these are Bernoulli trials. Thus, it is binomial dist'n

$$P_N(N_1) = \frac{N!}{N_1!\,(N-N_1)!}\left(\frac{V_1}{V}\right)^{N_1}\left(\frac{V_2}{V}\right)^{N-N_1}$$

$$P_N(N_1) = \frac{N!}{N_1!\,(N-N_1)!}\left(\frac{V_1}{V}\right)^{N_1}\left(\frac{V-V_1}{V}\right)^{N-N_1}$$

(c) $\bar{N}_1 = pN = \frac{V_1}{V}N$

$\bar{N}_2 = \frac{(V-V_1)}{V}N$

(d) $\frac{\sigma_1}{\bar{N}_1} = \frac{\sqrt{Npq}}{pN} = \sqrt{\frac{q}{p}}\frac{1}{\sqrt{N}} = \sqrt{\frac{V_2}{V_1}}\frac{1}{\sqrt{N}} = \sqrt{\frac{V-V_1}{V_1}}\frac{1}{\sqrt{N}}$

$\frac{\sigma_2}{\bar{N}_2} = \sqrt{\frac{V_1}{V_2}}\frac{1}{\sqrt{N}} = \sqrt{\frac{V_1}{V-V_1}}\frac{1}{\sqrt{N}}$

Problem 8 G & T Problem 3.51

Lets go over ideas leading to (3.169) ...
in order to show
$\beta = 0$ for $\bar{n} = 7/2$
$\beta = \infty$ for $\bar{n} = 1$
$\beta = -\infty$ for $\bar{n} = 6$
$\beta = -0.1746$ for $\bar{n} = 4$

Spose want to maximize
$f(x_1 \cdots x_N)$ subject to $g_j(x_1, \ldots, x_N) = 0$ $j = 1, \ldots, M$
$M < N$. Then

$$df = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} dx_i = 0$$ but dx_i are

not all indep... b/c

$$dg_j = \sum_{i=1}^{N} \frac{\partial g_j}{\partial x_i} dx_i = 0 \quad \text{for } j = 1, \ldots, M$$

This knocks out independence of M of the dx_i.
Our trick/technique is to write

$$df - \sum_{j=1}^{M} \lambda_j dg_j = 0 \quad \text{or} \ldots\ldots$$

$$\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} - \sum_{j=1}^{N} \lambda_j \frac{\partial g_j}{\partial x_i} \right) dx_i = 0$$

choose the M values of λ_j s.t. first M terms are individually zero. For remainder, $dx_{M+1}, \ldots dx_N$ can be individually varied so we have N-M eq's of form

$$\frac{\partial f}{\partial x_i} - \sum_{j=1}^{M} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = M+1, \ldots N$$

$$\frac{\partial f}{\partial P_i} - \sum_{j=1} \lambda_j \frac{\partial g_j}{\partial P_i} = 0$$

Problem 8 cont

Here is example: $N = 6$ $P_1, \dots, P_6$ on die

$M = 2$

$$f(P_1 \dots P_6) = S = -\sum_{i=1}^{6} P_i \ln P_i$$

$$g_1 = \sum_{i=1}^{6} P_i - 1$$

$$g_2 = \sum_{i=1}^{6} i P_i - \bar{n}$$

$\bar{n}$ is determined by, say, knowing you have a loaded die ...

$\bar{n} = 3.5$, maybe.

$$\therefore \frac{\partial f}{\partial P_i} = \frac{\partial S}{\partial P_i} = -(1 + \ln P_i)$$

$$\frac{\partial g_1}{\partial P_i} = 1$$

$$\frac{\partial g_2}{\partial P_i} = i$$

Thus for first two terms

$$-(1 + \ln P_1) - \alpha - \beta = 0$$

$$-(1 + \ln P_2) - \alpha - 2\beta = 0$$

$\alpha = \lambda_1$

$\beta = \lambda_2$

Prob 8 (cont again)

$\Rightarrow \alpha = \ln P_2 - 2\ln P_1 - 1$

$\beta = \ln P_1 - \ln P_2$

$\Rightarrow \ln P_1 = -1 - \alpha - \beta$

$\ln P_2 = -1 - \alpha - 2\beta$

Other prob's can be varied independently

eq. $\frac{\partial S}{\partial P_3} - \sum_{j=1}^{2} \lambda_j \frac{\partial g_j}{\partial P_3} = 0$

$\Rightarrow -(1 + \ln P_3) - \alpha - 3\beta = 0$

$\Rightarrow \ln P_3 = -1 - \alpha - 3\beta \qquad j=3$

also for $j = 4, 5, 6$

$\therefore \quad P_j = e^{-(1+\alpha)} e^{-\beta j}$

α from $\sum_j P_j = 1$

$\Rightarrow e^{-(1+\alpha)} = \frac{1}{\sum_i P_i}$

β from $\bar{n} = \sum_i i P_i$

Prob 8 cont yet again

$$\Rightarrow \bar{n} = \frac{e^{-\beta} + 2e^{-2\beta} + 3e^{-3\beta} + 4e^{-4\beta} + 5e^{-5\beta} + 6e^{-6\beta}}{e^{-\beta} + e^{-2\beta} + e^{-3\beta} + e^{-4\beta} + e^{-5\beta} + e^{-6\beta}}$$

Phew!

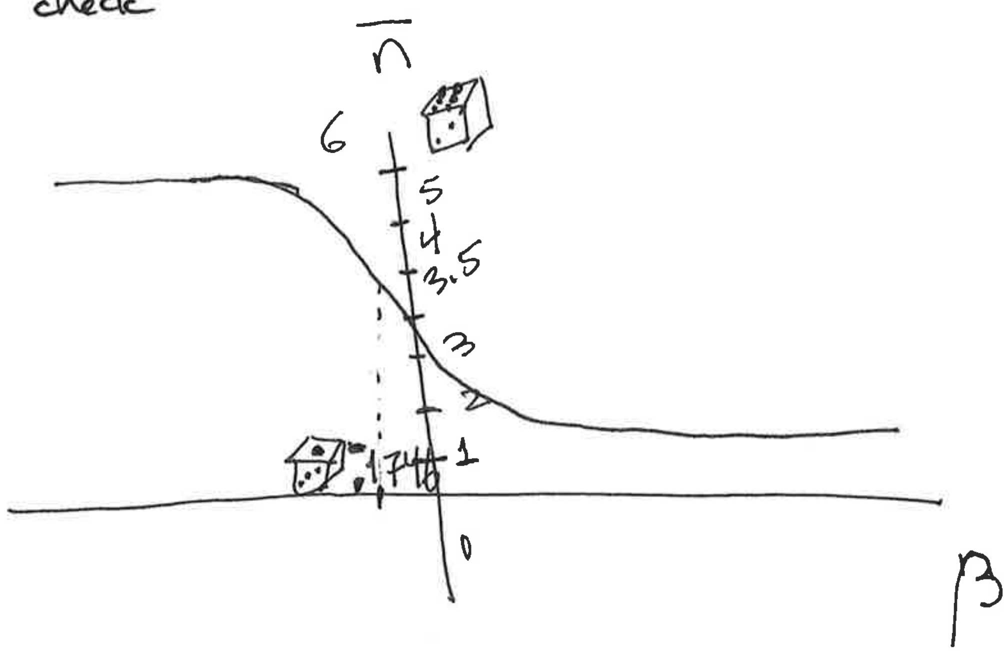
To the problem...

$\underline{\beta = 0}$ $\quad \bar{n} = \frac{21}{6} = 7/2$

$\underline{\beta = \infty}$ $\quad \bar{n} = 1$

$\underline{\beta = -\infty}$ $\quad \bar{n} = 6$

$\bar{n} = 4 \underset{\text{numerical check}}{\Longleftrightarrow} \beta = -0.1746$:)



Prob 8

$P_1 \cdots P_6$ $\quad N = 6$ vars

$M = 2$ constraints.

$g_1(P_1 \cdots P_6)$ is $\sum_i P_i - 1 = 0$

$g_2(P_1 \cdots P_6)$ is $\sum_j j P_j = \bar{n}$

choose

$f(P_1 \cdots P_6) = S = -\sum_i P_i \ln P_i$ large as possible

$$\boxed{\frac{\partial f}{\partial P_i} - \sum_{j=1}^{2M} \lambda_j \frac{\partial g_j}{\partial P_i} = 0}$$

$\lambda_1 = \alpha$

$\lambda_2 = \beta$

U will get

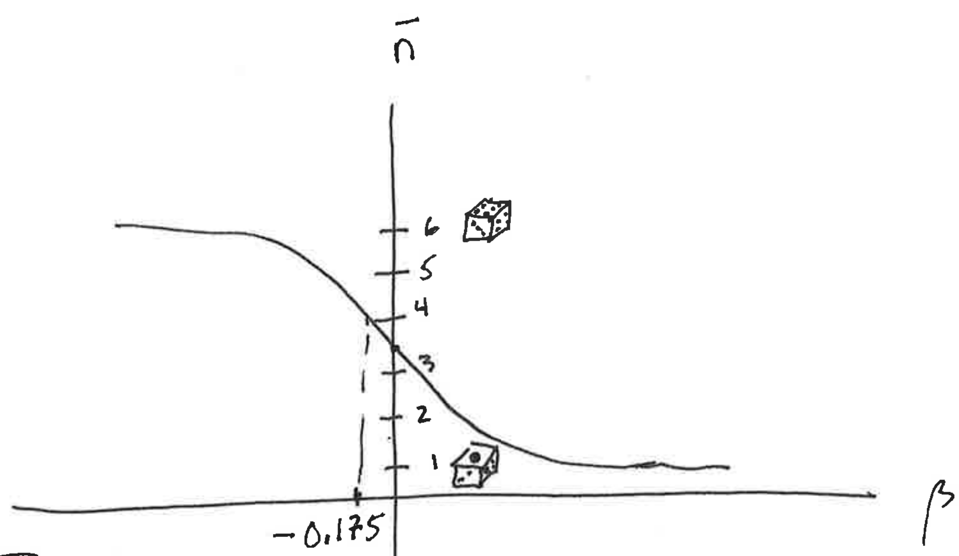
$$\underbrace{-(1 + \ln P_j)}_{-(1+\alpha)} - \alpha - \underbrace{j\beta}_{-\beta j} = 0 \quad \forall j$$

$\Rightarrow P_j = e \quad e^{-(1+\alpha)}$

$\sum_j P_j = 1 \Rightarrow e^{-(1+\alpha)} = \dfrac{1}{\sum_j P_j}$

β from $\bar{n} = \sum_j j P_j$

Prob 8



Sem 5

$$\boxed{\beta = \frac{1}{kT}}$$

Problem 9 G+T 3.33

Stirling's Approx.

(a) Mathematica can do a lot of digits. But a typical old school one does max n! of 170! = 7.257415615308 × 10^{306}

My iphone in fact tells me Error at this!

It can only do 103! = 9.9029007164 9e163

(b) cf. 3.102 $\ln N! \approx N\ln N - N$

+ 3.101 $\ln N! \approx N\ln N - N + \frac{1}{2}\ln(2\pi N)$

+ exact $\ln N!$

N	$N\ln N - N$	$N\ln N - N + \frac{1}{2}\ln(2\pi N)$	N!
5	3.047	4.770	4.787
10	13.025	15.096	15.104
20	39.914	42.331	42.335
50	145.601	148.476	148.477

(c) WTS: From 3.102, $\frac{d}{dx}\ln x! = \ln x$ for $x \gg 1$

$$\frac{d}{dx}\ln x! \approx \frac{d}{dx}(x\ln x - x) = \ln x + 1 - 1 = \ln x$$

QED