## Problem 5.10

(a) The forces on the two sides cancel. At the bottom, $B=\frac{\mu_{0} I}{2 \pi s} \Rightarrow F=\left(\frac{\mu_{0} I}{2 \pi s}\right) I a=\frac{\mu_{0} I^{2} a}{2 \pi s}$ (up). At the top, $B=\frac{\mu_{0} I}{2 \pi(s+a)} \Rightarrow F=\frac{\mu_{0} I^{2} a}{2 \pi(s+a)}$ (down). The net force is $\frac{\mu_{0} I^{2} a^{2}}{2 \pi s(s+a)}$ (up).
(b) The force on the bottom is the same as before, $\mu_{0} I^{2} a / 2 \pi s$ (up). On the left side, $\mathbf{B}=\frac{\mu_{0} I}{2 \pi y} \hat{\mathbf{z}}$;
$d \mathbf{F}=I(d \mathbf{l} \times \mathbf{B})=I(d x \hat{\mathbf{x}}+d y \hat{\mathbf{y}}+d z \hat{\mathbf{z}}) \times\left(\frac{\mu_{0} I}{2 \pi y} \hat{\mathbf{z}}\right)=\frac{\mu_{0} I^{2}}{2 \pi y}(-d x \hat{\mathbf{y}}+d y \hat{\mathbf{x}})$. But the $x$ component cancels the corresponding term from the right side, and $F_{y}=-\frac{\mu_{0} I^{2}}{2 \pi} \int_{s / \sqrt{3}}^{(s / \sqrt{3}+a / 2)} \frac{1}{y} d x$. Here $y=\sqrt{3} x$, so
$F_{y}=-\frac{\mu_{0} I^{2}}{2 \sqrt{3} \pi} \ln \left(\frac{s / \sqrt{3}+a / 2}{s / \sqrt{3}}\right)=-\frac{\mu_{0} I^{2}}{2 \sqrt{3} \pi} \ln \left(1+\frac{\sqrt{3} a}{2 s}\right)$. The force on the right side is the same, so the net force on the triangle is $\frac{\mu_{0} I^{2}}{2 \pi}\left[\frac{a}{s}-\frac{2}{\sqrt{3}} \ln \left(1+\frac{\sqrt{3} a}{2 s}\right)\right]$.


## Problem 5.14

(a) $\oint \mathbf{B} \cdot d \mathbf{l}=B 2 \pi s=\mu_{0} I_{\mathrm{enc}} \Rightarrow \mathbf{B}=\left\{\begin{array}{ll}\mathbf{0}, & \text { for } s<a ; \\ \frac{\mu_{0} I}{2 \pi s} \hat{\boldsymbol{\phi}}, & \text { for } s>a .\end{array}\right\}$
(b) $J=k s ; I=\int_{0}^{a} J d a=\int_{0}^{a} k s(2 \pi s) d s=\frac{2 \pi k a^{3}}{3} \Rightarrow k=\frac{3 I}{2 \pi a^{3}} . \quad I_{\mathrm{enc}}=\int_{0}^{s} J d a=\int_{0}^{s} k \bar{s}(2 \pi \bar{s}) d \bar{s}=$ $\frac{2 \pi k s^{3}}{3}=I \frac{s^{3}}{a^{3}}$, for $s<a ; I_{\mathrm{enc}}=I$, for $s>a$. So $\mathbf{B}=\left\{\begin{array}{ll}\frac{\mu_{0} I s^{2}}{2 \pi a^{3}} \hat{\phi}, & \text { for } s<a ; \\ \frac{\mu_{0} I}{2 \pi s} \hat{\phi}, & \text { for } s>a .\end{array}\right\}$

## Problem 5.32

(a) At the surface of the solenoid, $\mathbf{B}_{\text {above }}=0, \mathbf{B}_{\text {below }}=\mu_{0} n I \hat{\mathbf{z}}=\mu_{0} K \hat{\mathbf{z}} ; \hat{\mathbf{n}}=\hat{\mathbf{s}} ;$ so $\mathbf{K} \times \hat{\mathbf{n}}=-K \hat{\mathbf{z}}$. Evidently Eq. 5.76 holds. $\checkmark$
(b) In Eq. 5.69, both expressiions reduce to $\left(\mu_{0} R^{2} \omega \sigma / 3\right) \sin \theta \hat{\phi}$ at the surface, so Eq. 5.77 is satisfied. $\left.\frac{\partial \mathbf{A}}{\partial r}\right|_{R^{+}}=\left.\frac{\mu_{0} R^{4} \omega \sigma}{3}\left(-\frac{2 \sin \theta}{r^{3}}\right) \hat{\boldsymbol{\phi}}\right|_{R}=-\frac{2 \mu_{0} R \omega \sigma}{3} \sin \theta \hat{\boldsymbol{\phi}} ;\left.\quad \frac{\partial \mathbf{A}}{\partial r}\right|_{R^{-}}=\frac{\mu_{0} R \omega \sigma}{3} \sin \theta \hat{\boldsymbol{\phi}}$. So the left side of Eq. 5.78 is $-\mu_{0} R \omega \sigma \sin \theta \hat{\boldsymbol{\phi}}$. Meanwhile $\mathbf{K}=\sigma \mathbf{v}=\sigma(\boldsymbol{\omega} \times \mathbf{r})=\sigma \omega R \sin \theta \hat{\boldsymbol{\phi}}$, so the right side of Eq. 5.78 is $-\mu_{0} \sigma \omega R \sin \theta \hat{\phi}$, and the equation is satisfied.

## Problem 5.48

The total charge on the shaded ring is $d q=\sigma(2 \pi r) d r$. The time for one revolution is $d t=2 \pi / \omega$. So the current in the ring is $I=\frac{d q}{d t}=\sigma \omega r d r$. From Eq. 5.41, the magnetic field of

this ring (for points on the axis) is $d \mathbf{B}=\frac{\mu_{0}}{2} \sigma \omega r \frac{r^{\star}}{\left(r^{2}+z^{2}\right)^{3 / 2}} d r \hat{\mathbf{z}}$, and the total field of the disk is

$$
\begin{aligned}
\mathbf{B} & =\frac{\mu_{0} \sigma \omega}{2} \int_{0}^{R} \frac{r^{3} d r}{\left(r^{2}+z^{2}\right)^{3 / 2}} \hat{\mathbf{z}} . \quad \text { Let } u \equiv r^{2}, \text { so } d u=2 r d r \text {. Then } \\
& =\frac{\mu_{0} \sigma \omega}{4} \int_{0}^{R^{2}} \frac{u d u}{\left(u+z^{2}\right)^{3 / 2}}=\left.\frac{\mu_{0} \sigma \omega}{4}\left[2\left(\frac{u+2 z^{2}}{\sqrt{u+z^{2}}}\right)\right]\right|_{0} ^{R^{2}}=\frac{\mu_{0} \sigma \omega}{2}\left[\frac{\left(R^{2}+2 z^{2}\right)}{\sqrt{R^{2}+z^{2}}}-2 z\right] \hat{\mathbf{z}} .
\end{aligned}
$$

When $z \gg R$, the term in square brackets is

$$
\begin{aligned}
{[] } & =\frac{2 z^{2}\left(1+R^{2} / 2 z^{2}\right)}{z\left[1+(R / z)^{2}\right]-1 / 2}-2 z \approx 2 z\left[\left(1+\frac{R^{2}}{2 z^{2}}\right)\left(1-\frac{1}{2} \frac{R^{2}}{z^{2}}+\frac{3}{8} \frac{R^{4}}{z^{4}}\right)-1\right] \\
& \approx 2 z\left(1-\frac{R^{2}}{2 z^{2}}+\frac{3}{8} \frac{R^{4}}{z^{4}}+\frac{R^{2}}{2 z^{2}}-\frac{R^{4}}{4 z^{4}}-1\right)=2 z\left(\frac{1}{8} \frac{R^{4}}{z^{4}}\right)=\frac{R^{4}}{4 z^{4}},
\end{aligned}
$$

so

$$
\mathbf{B} \approx \frac{\mu_{0} \sigma \omega}{2} \frac{R^{4}}{4 z^{4}} \hat{\mathbf{z}}=\frac{\mu_{0} \sigma \omega R^{4}}{8 z^{3}} \hat{\mathbf{z}} .
$$

Meanwhile, from Eq. 5.88, the dipole field is

$$
\mathbf{B}_{\mathrm{dip}}=\frac{\mu_{0} m}{4 \pi r^{3}}(2 \cos \theta \hat{\mathbf{r}}+\sin \theta \hat{\boldsymbol{\theta}})
$$

and for points on the $+z$ axis $\theta=0, r=z, \hat{\mathbf{r}}=\hat{\mathbf{z}}$, so $\mathbf{B}_{\text {dip }}=\frac{\mu_{0} m}{2 \pi z^{3}} \hat{\mathbf{z}}$. In this case (Problem 5.37a) $m=\pi \sigma \omega R^{4} / 4$, so $\mathbf{B}_{\mathrm{dip}}=\frac{\mu_{0} \sigma \omega R^{4}}{8 z^{3}} \hat{\mathbf{z}}$, in agreement with the approximation.

## Problem 5.41

(a) If positive charges flow to the right, they are deflected down, and the bottom plate acquires a positive charge.
(b) $q v B=q E \Rightarrow E=v B \Rightarrow V=E t=v B t$, with the bottom at higher potential.
(c) If negative charges flow to the left, they are also deflected down, and the bottom plate acquires a negative charge. The potential difference is still the same, but this time the top plate is at the higher potential.

# Physics 112: Classical Electromagnetism, Fall 2013 Seminar 6: Solutions 

## Seminar Problems

Write strategies and sketch solutions for your own use during seminar:

1. Problems 5.20 (c) and (d) (p. 242). The answer to problem 5.20 (a) is $\rho=1.4 \times 10^{10} \mathrm{C} / \mathrm{m}^{3}$ and (b) is $v=9.1 \times 10^{-3} \mathrm{~cm} / \mathrm{s}$.
2. Problem 5.21 (p. 243).
3. Problem 5.26 (p. 248). In discussing this problem we will discuss why some approaches for calculating vector potential do not work for this problem, and what approaches do work (Griffiths says "by whatever means you can think of..." because some approaches don't work).
4. Problem 5.56 (p. 262).

## Problems:

Warmup assignment (due Monday at 4:45 p.m.): Griffiths problems 5.10(a), 5.14(b), and 5.32.

1. Steady state: Show that if the current is steady state (i.e., all partial time derivatives vanish) then $d I / d \ell=0$, where $\ell$ lies along the wire carrying the current.

## Solution:

There are several ways of solving this problem; here are a few:
2. Magnetic field of a rotating disk: 5.48 (connects to a classic experiment)
3. Hall effect: 5.41 (this describes how we tell the sign of the charge carriers)
4. An better approximation to the solenoid
A. Find the magnetic field along the axis of a solenoid: Problem 5.11.

Solution: From Ex. 5.6 we know that the magnetic field a distance $z$ above the center of a circular loop of radius $R$ is

$$
\begin{equation*}
B(z)=\frac{\mu_{0} I}{2} \frac{R^{2}}{\left(R^{2}+z^{2}\right)^{3 / 2}} \tag{1}
\end{equation*}
$$



Figure 1: The coordinates for the solenoid.

Let $n$ denote the number of loops per length so that the number of loops in a small length $d z$ is

$$
\begin{equation*}
d N=n d z \tag{2}
\end{equation*}
$$

then we have

$$
\begin{equation*}
B(z)=\int d B=\frac{\mu_{0} I n}{2} \int_{z-L / 2}^{z+L / 2} \frac{1}{\left(R^{2}+z^{2}\right)^{3 / 2}} d z=\alpha\left[\frac{z_{+}}{\sqrt{R^{2}+z_{+}^{2}}}-\frac{z_{-}}{\sqrt{R^{2}+z_{-}^{2}}}\right] \tag{3}
\end{equation*}
$$

where $\alpha \equiv \operatorname{In} \mu_{0} / 2$ and $z_{ \pm}=z \pm L / 2$.
B. Now we will use some techniques described in Chapter 3 in order to (approximately) determine the magnetic field off of the axis. Inside of the solenoid the equation of magnetostatics are

$$
\begin{array}{r}
\vec{\nabla} \cdot \vec{B}=0 \\
\vec{\nabla} \times \vec{B}=0 \tag{5}
\end{array}
$$

The second equation means that we can define a 'magnetic scalar potential', $\phi_{M}$, for which

$$
\begin{equation*}
\vec{B}=-\vec{\nabla} \phi_{M} \tag{6}
\end{equation*}
$$

In addition to this, since $\vec{\nabla} \cdot \vec{B}=0$, this scalar potential satisfies the Laplace equation:

$$
\begin{equation*}
\nabla^{2} \phi_{M}=0 \tag{7}
\end{equation*}
$$

Show that the magnetic scalar potential takes the form

$$
\begin{equation*}
\phi_{M}(r, \theta)=\sum_{\ell=0}^{\infty} a_{\ell} r^{\ell} P_{\ell}(\cos \theta) . \tag{8}
\end{equation*}
$$

For points along the axis we have

$$
\begin{equation*}
\phi_{M}(z)=\sum_{\ell=0}^{\infty} a_{\ell} z^{\ell} \tag{9}
\end{equation*}
$$

Solution:

Since the 'magnetic scalar potential' follows the Laplace equation we know that we can use the separation of variables to solve it. In spherical coordinates, with azimuthal symmetry, we can appeal to Eq. (3.65) to write in general

$$
\begin{equation*}
\phi_{M}(r, \theta)=\sum_{\ell=0}^{\infty}\left(a_{\ell} r^{\ell}+\frac{b_{\ell}}{r^{1+\ell}}\right) P_{\ell}(\cos \theta) \tag{10}
\end{equation*}
$$

In this case we are describing the magnetic field within the solenoid, so we require that the magnetic scalar potential must be well-behaved at $r=0-$ this means $B_{\ell}=0$. Therefore the magnetic vector potential takes the form

$$
\begin{equation*}
\phi_{M}(r, \theta)=\sum_{\ell=0}^{\infty} a_{\ell} r^{\ell} P_{\ell}(\cos \theta) \tag{11}
\end{equation*}
$$

Finally, for points along the axis of the solenoid we have $\cos \theta=1$ and you can look up (or check using Mathematica) that $P_{\ell}(1)=1$ for all values of $\ell$. Therefore on axis:

$$
\begin{equation*}
\phi_{M}(z)=\sum_{\ell=0}^{\infty} a_{\ell} z^{\ell} \tag{12}
\end{equation*}
$$

C. From part (A) show that the magnetic scalar potential along the $z$-axis is given by

$$
\begin{equation*}
\phi_{M}(z)=-\alpha\left[\left(R^{2}+z_{+}^{2}\right)^{1 / 2}-\left(R^{2}+z_{-}^{2}\right)^{1 / 2}\right], \tag{13}
\end{equation*}
$$

where

$$
\begin{align*}
\alpha & \equiv \frac{\mu_{0} N I}{4 \pi}  \tag{14}\\
z_{ \pm} & \equiv z \pm L / 2 \tag{15}
\end{align*}
$$

Expand Eq. (13) in a Taylor series around $z=0$ up to third order and match the coefficients with the expansion in Eq. (9) to find $a_{1}, a_{2}$, and $a_{3}$.

## Solution:

As was pointed out by a few solutions, the constant $\alpha$ as written above is wrong. Lets see what it should be.

By definition, the magnetic scalar potential is related to the magnetic field through

$$
\begin{equation*}
\vec{B}=-\vec{\nabla} \phi_{M} \rightarrow B_{z}(z)=-\left.\frac{\partial \phi_{M}}{\partial z}\right|_{x=y=0} \tag{16}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\phi_{M}(z, 0,0)=-\int B_{z}\left(z^{\prime}\right) d z^{\prime} \tag{17}
\end{equation*}
$$

Computing this integral we get

$$
\begin{equation*}
\phi_{M}(z, 0,0)=-\frac{I n \mu_{0}}{2}\left(\sqrt{R^{2}+z_{+}}-\sqrt{R^{2}+z_{-}}\right) . \tag{18}
\end{equation*}
$$

Therefore, $\alpha \equiv \frac{I n \mu_{0}}{2}$, as before.
D. Use your approximate solution for $\phi_{M}(r, \theta)$ to write down approximate expressions for $B_{r}, B_{\theta}$. Transforming to a cylindrical basis we use the equations

$$
\begin{align*}
& B_{s}=B_{r} \sin \theta+B_{\theta} \cos \theta,  \tag{19}\\
& B_{z}=B_{r} \cos \theta-B_{\theta} \sin \theta, \tag{20}
\end{align*}
$$

and

$$
\begin{align*}
& s=r \sin \theta  \tag{21}\\
& z=r \cos \theta \tag{22}
\end{align*}
$$

Take the ratio $B_{s} / B_{z}$. Identify which characteristics of the solenoid determine the extent to which the magnetic field is uniform within the coils (as described in Example 5.9).

## Solution:

We can now expand our expression for the magnetic scalar potential along the axis and match the expansion with the coefficients in Eq. (12).

Expanding Eq. (??) to third order in $z$ we have

$$
\begin{equation*}
\phi_{M}(z, 0,0) \simeq-\frac{2 L \alpha}{\sqrt{L^{2}+4 R^{2}}} z+\frac{16 L R^{2} \alpha}{\left(L^{2}+4 R^{2}\right)^{5 / 2}} z^{3} \tag{23}
\end{equation*}
$$

Therefore we have

$$
\begin{align*}
& a_{1}=-\frac{2 L \alpha}{\sqrt{L^{2}+4 R^{2}}}  \tag{24}\\
& a_{2}=\frac{16 L R^{2} \alpha}{\left(L^{2}+4 R^{2}\right)^{5 / 2}} \tag{25}
\end{align*}
$$

This, in turn, tells us that

$$
\begin{equation*}
\phi_{M}(r, \theta) \simeq-\frac{2 L \alpha}{\sqrt{L^{2}+4 R^{2}}} r \cos \theta+\frac{16 L R^{2} \alpha}{\left(L^{2}+4 R^{2}\right)^{5 / 2}} r^{3} \frac{1}{2}(5 \cos \theta-3 \cos \theta) \tag{26}
\end{equation*}
$$

We are now able to take the gradient of this to determine the form that the magnetic field takes:

$$
\begin{align*}
B_{r} & =\frac{2 L \alpha\left(L^{2} 4 R^{2}\right)^{2} \cos \theta-3 r^{2} R^{2}(3 \cos \theta-5 \cos 3 \theta)}{\left(L^{2}+4 R^{2}\right)^{5 / 2}}  \tag{27}\\
B_{\theta} & =-\frac{2 L \alpha\left(L^{4} 8 L^{2} R^{2}-18 r^{2} R^{2}+16 R^{4}-30 r^{2} R^{2} \cos 2 \theta\right) \sin \theta}{\left(L^{2}+4 R^{2}\right)^{5 / 2}} \tag{28}
\end{align*}
$$

It is straightforward (and using Mathematic, it is trivial) to convert this to a cylindrical basis:

$$
\begin{align*}
B_{s} & =\frac{48 L r^{2} R^{2} \alpha \cos \theta \sin \theta}{\left(L^{2}+4 R^{2}\right)^{5 / 2}}  \tag{29}\\
B_{z} & =\frac{2 L \alpha\left(L^{4}+8 L^{2} R^{2}-6 r^{2} R^{2}+16 R^{4}-18 r^{2} R^{2} \cos 2 \theta\right.}{\left(L^{2}+4 R^{2}\right)^{5 / 2}} \tag{30}
\end{align*}
$$

Finally, we can express this answer in cylindrical coordinates

$$
\begin{align*}
B_{s} & =\frac{48 L R^{2} \alpha}{\left(L^{2}+4 R^{2}\right)^{5 / 2}} s z  \tag{31}\\
B_{z} & =\frac{2 L\left(L^{4}+8 L^{2} R+4 R^{2}\left(4 R^{2}+3 s^{2}-6 z^{2}\right)\right) \alpha}{\left(L^{2}+4 R^{2}\right)^{5 / 2}} \tag{32}
\end{align*}
$$

The usual assumption about the magnetic field inside of a the solenoid is that it is uniform and points in along the axis. These equations allow us to see how good of an approximation this is! Comparing the z-component of the magnetic field at the center of the solenoid $(z=s=0)$ to the z-component off of the axis we have

$$
\begin{equation*}
\frac{B_{z}(s, z)}{B_{z}(0,0)}-1 \simeq \frac{12 R^{2}}{L^{4}}\left(s^{2}-2 z^{2}\right) \lesssim \frac{12 R^{2}}{L^{2}} \tag{33}
\end{equation*}
$$

where we have expanded the result to leading order in the limit that $R \ll L$ - i.e., the solenoid is long and skinny. Therefore, if $R=0.1 L$ the magnitude of the $z$-component of the magnetic field varies by less than $0.1 \%$ along the length of the solenoid.

In addition to this we can see the relative size of the $z$-component and $s$-component of the magnetic field as we move off of the axis of the solenoid. Again, the usual expression assumes that $B_{s}=0$ :

$$
\begin{equation*}
\frac{B_{s}(s, z)}{B_{z}(s, z)} \simeq 24 \frac{R^{2}}{L^{4}} s z \lesssim 24\left(\frac{R}{L}\right)^{3} \tag{34}
\end{equation*}
$$

Therefore, for $R / L=0.1$ the $s$-component of the magnetic field is only $0.01 \%$ of the $z$-component within the solenoid.

In the end, this problem has shown that the approximation that the magnetic field within a solenoid is directed along the $z$-axis and is uniform within the solenoid is a very good approximation whenever $R / L \ll 1$.

## 5. Photon mass.

It is possible that the photon (i.e., electromagnetic waves) carry mass. Mathematically this would imply that the Poisson equation which determines the electrostatic potential and the magnetic vector potentials are modified from their usual forms:

$$
\begin{align*}
\nabla^{2} V & =-\rho / \epsilon_{0}  \tag{35}\\
\nabla^{2} \vec{A} & =-\mu_{0} \vec{J} \tag{36}
\end{align*}
$$

to

$$
\begin{align*}
\nabla^{2} V-\mu^{2} V & =-\rho / \epsilon_{0}  \tag{37}\\
\nabla^{2} \vec{A}-\mu^{2} \vec{A} & =-\mu_{0} \vec{J} \tag{38}
\end{align*}
$$

The relationship between these potentials and the fields remains unchanged, namely:

$$
\begin{align*}
\vec{E} & =-\vec{\nabla} V  \tag{39}\\
\vec{B} & =\vec{\nabla} \times \vec{A} \tag{40}
\end{align*}
$$

A. From these equations, what are the units of the photon's mass? If this seems strange, remember that in quantum mechanics the wavelength associated with a moving particle is $\lambda=h / p$. If the photon has a very small mass then it will move at a speed close to the speed of light, $c$. Use this to write an expression for the mass of the photon in the usual units.

## Solution:

The de Broglie wavelength is

$$
\begin{equation*}
\lambda=\mu=\frac{h}{p}=\frac{h}{m_{\gamma} c} \rightarrow m_{\gamma}=\frac{h}{c \mu} \tag{41}
\end{equation*}
$$

B. The integral solution to the new equation for the electrostatic potential and the magnetic vector potential can be written as

$$
\begin{align*}
V & =\frac{1}{4 \pi \epsilon_{0}} \int \frac{\rho\left(\vec{r}^{\prime}\right)}{\imath} e^{-\mu\rangle} d \tau^{\prime}  \tag{42}\\
\vec{A} & =\frac{\mu_{0}}{4 \pi} \int \frac{\vec{J}\left(\vec{r}^{\prime}\right)}{\imath} e^{-\mu \imath} d \tau^{\prime} \tag{43}
\end{align*}
$$

Write down the electric field due to a point charge in this new theory.

## Solution:

A point charge has a charge density given in terms of the Dirac delta function

$$
\begin{equation*}
\rho\left(\vec{r}^{\prime}\right)=\frac{q}{4 \pi} \delta^{(3)}\left(\vec{r}^{\prime}\right) \tag{44}
\end{equation*}
$$

so that the electrostatic potential in this theory becomes

$$
\begin{equation*}
V(\vec{r})=\frac{q}{4 \pi \epsilon_{0}} e^{\mu \vec{r}} \tag{45}
\end{equation*}
$$

C. Now consider the vector potential produced by a linear current loop, with current $I$. Integrating over the loop we can write

$$
\begin{equation*}
\vec{A}=\frac{\mu_{0} I}{4 \pi} \int \frac{e^{-\mu \imath}}{\imath} d \overrightarrow{\ell^{\prime}} \tag{46}
\end{equation*}
$$

Using the identity in Problem 1.61e show that this can be written as

$$
\begin{equation*}
\vec{A}=-\frac{\mu_{0} I}{4 \pi} \int e^{-\mu \imath}\left(\frac{1}{\imath^{2}}+\frac{\mu}{\imath}\right) \hat{\imath} \times d \vec{a}^{\prime} \tag{47}
\end{equation*}
$$

Now, if we assume that we are very far away from the current loop, $\vec{\imath} \simeq \vec{r}$ and the vector potential becomes

$$
\begin{equation*}
\vec{A} \approx-\frac{\mu_{0}}{4 \pi} e^{-\mu r}\left(\frac{1}{r^{2}}+\frac{\mu}{r}\right) \hat{r} \times I \int d \vec{a}^{\prime}=\frac{\mu_{0}}{4 \pi} e^{-\mu r}\left(\frac{1}{r^{2}}+\frac{\mu}{r}\right)(\vec{m} \times \hat{r}) \tag{48}
\end{equation*}
$$

## Solution:

We are starting from

$$
\begin{equation*}
\vec{A}=\frac{\mu_{0} I}{4 \pi} \int \frac{e^{-\mu \imath}}{\imath} d \overrightarrow{\ell^{\prime}} \tag{49}
\end{equation*}
$$

The identity in 1.61 e states

$$
\begin{equation*}
\int_{\mathcal{S}} \vec{\nabla} T \times d \vec{a}=-\oint_{\mathcal{P}} T d \vec{\ell} \tag{50}
\end{equation*}
$$

Taking the divergence of $e^{-\mu \nu} / \imath$ we can then write

$$
\begin{equation*}
\vec{A}=-\frac{\mu_{0} I}{4 \pi} \int e^{-\mu \imath}\left(\frac{1}{\imath^{2}}+\frac{\mu}{\imath}\right) \hat{\imath} \times d \vec{a}^{\prime} . \tag{51}
\end{equation*}
$$

D. Taking the curl of the vector potential it is straightforward (but VERY tedious)- but, once it is taken we have the following expression for the magnetic field:

$$
\begin{equation*}
\vec{B}_{\mathrm{dip}}=\frac{\mu_{0}}{4 \pi} e^{-\mu r}[(3+\mu r[3+\mu r])(\hat{r} \cdot \vec{m}) \hat{r}-(1+\mu r[1+\mu r]) \vec{m}] \tag{52}
\end{equation*}
$$

Verify that we regain the usual expression for the magnetic field of a dipole when $\mu=0$.

## Solution:

The standard expression in Maxwell's theory is

$$
\begin{equation*}
\vec{B}_{\mathrm{dip}}=\frac{\mu_{0}}{4 \pi}[3(\vec{m} \cdot \hat{r}) \hat{r}-\vec{m}] . \tag{53}
\end{equation*}
$$

When $\mu=0$ it is clear that we regain the usual expression.
E. Imagine we have a magnetic dipole and we measure the magnetic field at a point along the direction of the dipole moment as well as at a point which is perpendicular to the direction of the dipole moment. Write down an expression for the ratio of the magnitude of the magnetic field at those two points.

At the pole we have $\hat{r} \cdot \vec{m}=m$ and

$$
\begin{equation*}
\left|B_{\mathrm{dip}}^{\text {pole }}\right|=\frac{\mu_{0}}{4 \pi} e^{-\mu r}[(3+\mu r[3+\mu r]) m(1+\mu r[1+\mu r]) m]=\frac{1}{2} \frac{\mu_{0}}{4 \pi} e^{-\mu r} m(1+\mu r) . \tag{54}
\end{equation*}
$$

On the equator we have $\hat{r} \cdot \vec{m}=0$ and

$$
\begin{equation*}
\left|B_{\mathrm{dip}}^{\text {equator }}\right|=-f r a c \mu_{0} 4 \pi e^{-\mu r} m\left(1+\mu r+\mu^{2} r^{2}\right) \tag{55}
\end{equation*}
$$

Taking the ratio of the two we have

$$
\begin{equation*}
\frac{B_{\mathrm{di}}^{\text {pole }}}{B_{\mathrm{dip}}^{\text {equator }}}=\frac{2(1+\mu r)}{1+\mu r+\mu^{2} r^{2}} \tag{56}
\end{equation*}
$$

We can use these measurements to place a constraint on the value of $\mu$. To place the most restrictive constraint, will we want to make $r$ as large as we can or as small as we can?

## Solution:

Let us first assume that the data we collect will be consistent with Maxwell's theory in which $\mu=0$. Then, the data will allow us to place an upper limit on $\mu$. In Maxwell's theory we expect $\frac{B_{\text {dip }}^{\text {dole }}}{B_{\text {dip }}^{\text {potor }}}=2$. Data agrees with this to one part in 1000 . Given that the radius of the earth is $R_{\oplus}=6,400 \mathrm{~km}$ this places a constraint on the mass of the photon $\mu<9.8 \times 10^{8} \mathrm{~cm}^{-1}$. ${ }^{1}$

[^0]
[^0]:    ${ }^{1}$ Goldhaber and Nieto, Phys. Rev. Lett. 21, 567 (1968)

