Physics 112: Classical Electromagnetism, Fall 2013
Seminar 4

Material summary:

• In this chapter we explore the unique properties of the Poisson equation, which dictates the behavior of the electrostatic potential in the presence of a charge density ρ:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{1}{\epsilon_0}\rho. \tag{1}$$

Griffiths does this by first focusing on the case when the charge density is zero: $\nabla^2 V = 0$. This equation is known as the Laplace equation and has some beautiful properties.

The most important property is known as the *mean value property*:

$$V(\vec{r}) = \frac{1}{4\pi R} \oint_{\text{sphere}} V da. \tag{2}$$

This property allows us to conclude (in Sec. 3.1.5) that "The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$ of $\mathcal{V}$." There are other uniqueness theorems that follow from this one. The rest of the chapter explores the amazing consequences of this theorem by presenting various ways of solving both the Poisson equation and the Laplace equation. *Note*: make sure to know the difference between the Poisson and Laplace equations! They are related but different!

• The central theme of this chapter is the potential on a boundary surface $\mathcal{S}$ of some volume $\mathcal{V}$: make sure to understand what these concepts really mean! Draw some pictures of volumes and boundaries.

• The ability to use the method of images (Sec. 3.2) and separation of variables (Sec. 3.3) depends on the uniqueness theorems. The method of images is not as widely used as the separation of variables in other areas of physics. Think about how strange the separation of variables is: we are looking for a function $V(x, y, z)$ and then we *assume* that $V(x, y, z) = X(x)Y(y)Z(z)$! That is an amazing assumption if you think about it– *most* scalar fields cannot be written in that form! Our justification is the uniqueness theorem related to the Laplace and Poisson equations.

• From this, we arrive at the multipole expansion. This is probably one of the most useful parts of the chapter: the idea that we can write the potential (or electric field) due to some charge distribution as a sum of terms of decreasing size comes up again and again throughout physics. For instance: regardless of how complicated a charge distribution is, if it has a total non-zero charge, then far away it will appear as a point charge. We know this must be the case intuitively, and this section puts mathematics where our intuition is.

Reading:

- Griffiths: Chapter 3.
- Notes on buckling.

  These notes describe the physics of a buckling card. There are more details in these notes than you will need for class (but if you want, the details are there). The beginning talks about how the non-linear nature of buckling causes a failure of the types of uniquness theorems that we have in electromagnetism. In addition, the non-linearity leads to a very interesting degeneracy when it comes to the minimum of the energy of the system.

- Maxwell's preface to his *Treatise on Electromagnetism*.

  An introduction from the great Maxwell himself. Pay attention to his focus on "measurable quantities" and his deep respect for experiment and observation.

Snacks!: Stefan

Presentations:

- **Alex:** Problem 3.2 and the practical consequences of Earnshaw's theorem.

- **Chris:** Compare and contrast separation of variables in Cartesian, spherical, and cylindrical coordinates (see Problem 3.24). Be sure to explain *why* we can even use separation of variables in the first place.

- **Adrian:** Identify the core of the multipole expansion and answer the question: "what relationship (if any) does the multipole expansion have with the uniqueness theorems and separation of variables?"

In answering this question, be sure to present a (simple) derivation of Eq. (3.94),

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha). \tag{3}$$

- **Mike:** How would the differential equations of electrostatics change if all charge came in the form of pure electric dipoles? (*Hint:* you will have to use the result of Problem 3.36 and use Mathematica to evaluate the divergence and curl of the electric field.) Does your answer change if the dipoles are not pure?

Problems:

1. Please do the following problems from Chapter 3:

The uniqueness theorems: 3.6; how does the second uniqueness theorem prove that the capacitance of a system of conductors is purely geometric? Is the second uniqueness theorem completely unrelated to the first uniqueness theorem?

Earnshaw's theorem: 3.2

Averages: 3.47 (this result will prove invaluable when we look into the question "why is the sky blue?")

Method of images: 2.52 and 3.12

Separation of variables: 3.16, 3.21, 3.24

Green's reciprocity theorem: 3.50; what basic aspects of electrostatics underlies this theorem?

Also, use this theorem to prove the following, related to Maxwell's test of Coulomb's law:

Starting with two nested conductors, (not necessarily spherical nor concentric) we charge them up to a common potential V_0; call this initial state I. We then insulate the two conductors from each other, and ground the outer conductor; call this final state II. Show that Green's reciprocity theorem allows us to conclude that

$$\frac{V_{II,\ \text{inner}}}{V_0} = \frac{Q_{II,\ \text{outer}}}{Q_{I,\ \text{inner}}} + 1. \tag{4}$$

Multipoles: 3.36, 3.52 (what is the relationship, if any, to the *moment of inertia*?)

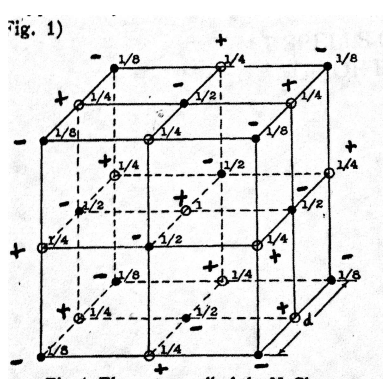

Fig. 1. Elementary cell of the NaCl-type.

Figure 1: Problem 4: the elementary cube of NaCl.

2. *Multipole moment of a salt crystal.* In Fig. 1 you see the ion distribution in a salt (NaCl) crystal. The numbers indicate the fraction of a charge at each point in the lattice associated with this cube (the rest of the charge is associated with neighboring cubes).

Following the progression presented in chapter 3's Fig. 3.27, make a guess as to how the electrostatic potential of this charge distribution depends on distance.

Check your answer using the Mathematica notebook on Moodle under Seminar 4. If you guess was incorrect, revise it in order to explain the radial dependence of the electrostatic potential.

How does knowledge of this radial dependence affect how we view NaCl's crystal structure?

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3. *Equipotentials of a conductor in a uniform electric field.*

A. Plot the equipotentials for the electrostatic potential derived in example 3.8. Be sure to understand how this potential was derived!

Where is the potential zero?

B. Express the dipole moment of the conducting sphere in terms of its volume and the (uniform) external electric field. We'll see that this quantity will come up again in Chapter 4.

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4. *The uniqueness theorem for other theories.*

In this chapter you've been introduced to the amazing properties of the Laplace equation. You have also seen how these properties allow us to make remarkable headway when trying to solve the Laplace equation. In this problem you will see equations that apply to very different physical situations and try your hand at showing that they too have solutions which are unique when specified on a boundary.

A. Elastostatics. In elastostatics we are interested in describing how a solid object, like a slab of wood, responds to external stresses. In doing this, we describe the flexing of the material by a displacement vector. If, when unstressed, a small piece of the material is at a location $\vec{x}$, then under strain it will move to a location $\vec{x}' = \vec{x} + \vec{\xi}(\vec{x})$: $\vec{\xi}(\vec{x})$ is the *displacement vector.*

In the theory of *linear* elastostatics, the displacement vector is the solution to the equation

$$\left(K + \frac{1}{3}\mu\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{\xi}) + \mu\nabla^2\vec{\xi} = 0, \tag{5}$$

where K and μ are constants associated with the properties of the material that are under strain. The constant K is known as the *bulk modulus* and μ is the *shear modulus.* These constants generalize the spring constant k in Hooke's law, $\vec{F} = -k\vec{x}$, to three dimensions.

Show that the above equation implies

$$\nabla^2(\vec{\nabla}\cdot\vec{\xi}) = 0, \tag{6}$$

$$\nabla^2\nabla^2\vec{\xi} = 0. \tag{7}$$

Note: the usual Laplacian, $\nabla^2 V$, is different than the *vector* Laplacian $\nabla^2\vec{\xi}$ [1]. Look at property (11) in the vector identities in the front cover of Griffiths. (*Hint*: show the first equation is true and that will help you show that the second equation is true.)

[1] In Cartesian coordinates the vector Laplacian looks like $\nabla^2\vec{\xi} = \frac{\partial^2\xi_x}{\partial x^2}\hat{x} + \frac{\partial^2\xi_y}{\partial y^2}\hat{y} + \frac{\partial^2\xi_z}{\partial z^2}\hat{z}$. However, in other coordinate systems it is much more complicated!

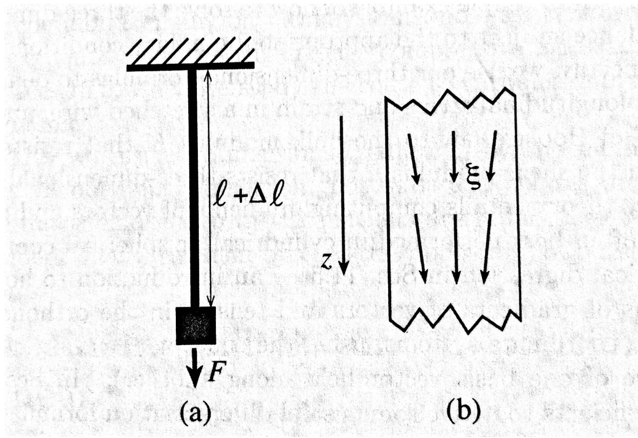

Figure 2: The displacement vector when an object is strechted.

The second equation shows that the displacement vector is *biharmonic* field. An field that satisfies Laplace's equation, $\nabla^2 V = 0$, is said to be *harmonic*.

What aspects of $\vec{\xi}$ need to be specified on the boundary of a volume in order to have a unique solution of $\vec{\xi}$ in the interior of that boundary?

B. Stokes flow. A fluid with a large viscosity (like molasses) is a Stokes flow and follows the set of equations

$$\begin{aligned} \vec{\nabla} P - \mu \nabla^2 \vec{v} &= 0, & (8) \\ \vec{\nabla} \cdot \vec{v} &= 0, & (9) \end{aligned}$$

where μ is the viscosity of the fluid, P is the pressure in the fluid (which can change from point to point), and $\vec{v}$ is the velocity.

We want to prove a uniqueness theorem for this set of equations just as Griffiths proved it for electrostatics.

Just as in electrostatics, we suppose that the fluid velocity is known on some boundary, $\vec{v}_s$ (note: the velocity can be different at different points on the boundary!). We now suppose that we have two solutions to the above equations, $\vec{v}_1$ and $\vec{v}_2$, both of which equal $\vec{v}_s$ on the boundary. We now want to show that this implies $\vec{v}_1 = \vec{v}_2$.

To do this we define a new velocity $\vec{v} \equiv \vec{v}_1 - \vec{v}_2$ which, by construction, is zero on the boundary (why?).

• Argue that within the volume

$$\vec{v} \cdot \left(\vec{\nabla} P - \mu \nabla^2 \vec{v} \right) = 0. \tag{10}$$

Therefore we know that when integrated over the volume:

$$\int_{\mathcal{V}} \vec{v} \cdot \left(\vec{\nabla} P - \mu \nabla^2 \vec{v} \right) dV = 0. \tag{11}$$

• Write out this integrand in index notation and show that

$$0 = \int_{\mathcal{V}} \vec{v} \cdot \left(\vec{\nabla} P - \mu \nabla^2 \vec{v} \right) dV = \int_{\mathcal{V}} \left[\partial_i (v_i P) - \mu \partial_j (v_i \partial_j v_i) + \mu (\partial_j v_i)(\partial_j v_i) \right] dV \tag{12}$$

• Explain why the first two terms on the left-hand side of Eq. (12) vanishes so that we can rewrite the above equation as

$$\int_{\mathcal{V}} (\partial_j v_i)(\partial_j v_i) dV = 0, \tag{13}$$

Since the integrand is positive definite this term can only vanish if $\partial_j v_i = 0$ within the volume. Show that this can only be true if v_i is constant. Since we know $\vec{v}$ is zero on the boundary and is constant within the volume, it must be zero throughout the volume.

<u>What to take away from this exercise</u>: any time you encounter a new set of equations which describes a physical phenomena **stop and ask** "what is required in order to produce a unique solution to these equations?" The answer to this question will determine what methods you are allowed to use in order to solve the equations– i.e., if a unique solution *is not* purely specified by the value on a boundary then the separation of variables will not, in general, work!
