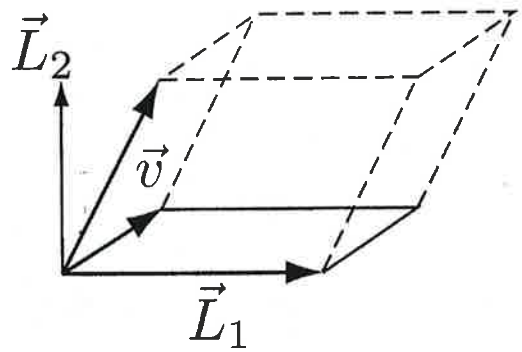
Solution to an extension of problem 1

1. *Dot and cross your $\vec{A}$s and $\vec{B}$s.*

Restate the problem so that everyone knows what you are talking about.



In a time dt what mass of fluid (of density ρ and flowing with velocity $\vec{v}$) flows through the surface defined by vectors $\vec{L}_1$ and $\vec{L}_2$ in the above figure?

Use this to show that the flux of fluid (mass-per time-per area) is

$$\mathcal{F} \equiv \frac{dM}{Adt} = \rho(\vec{v} \cdot \hat{n}). \tag{1}$$

State clearly what you are solving

As we discussed in seminar, with the above expression for the flux, we can start to answer some interesting questions about everyday science.

In particular, we discussed the tapering of a stream of water as it flows from a spigot, as shown below in Fig. 1. We can use our expression for the flux in order to derive an expression which predicts how the width of the stream will change with distance from the opening of the spigot.

First we note that if the rate of fluid flow, dM/dt, changes along the length of the stream of water then that would lead to a build up of the water along the stream with time, which we can see does not happen. This means that

$$\frac{dM}{dt} = \text{const.}, \tag{2}$$

along the length of the stream.

Refer to figures so people can know to look at them

As indicated in Fig. 1, we will denote the vertical height at the opening of the spigot as $h = 0$ and take the positive direction of h as downward in the figure. We will further denote the cross sectional area of the stream at $h = 0$ as A_0 and the velocity at that point as v_0. Therefore as we move along the stream we have

$$\rho A(h)v(h) = \rho A_0 v_0 \to \frac{A(h)}{A_0} = \frac{v_0}{v(h)}. \tag{3}$$

Be clear about what quantities depend on

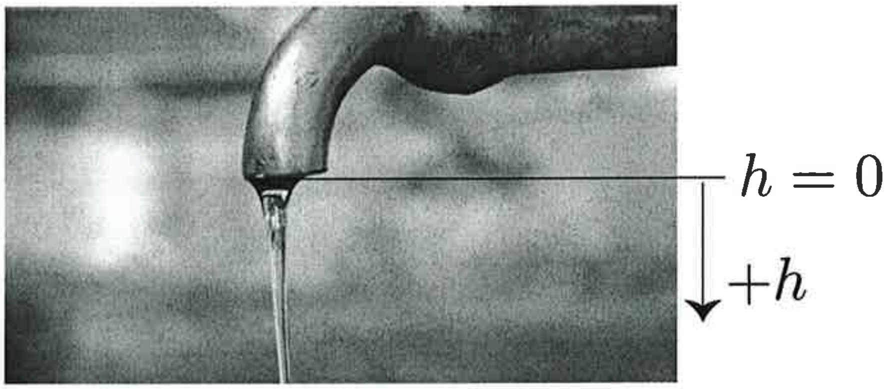


Figure 1: Water flowing from a spigot.

As much as possible, describe the physics in words before the equations

In order to determine how the velocity of the stream changes with height we need to understand what forces are acting on the water. First among the forces is gravity. The water also experiences a certain amount of surface tension, but we will assume that this force is insignificant when compared to the force of gravity.

Therefore, each small bit of water, of mass dm, accelerates at a constant rate $g \simeq 10$ m/s^2. We can determine $v(h)$ by using the standard equations of kinematics, but it turns out that it is easier to determine this relationship by considering energy conservation.

Indicate if there are multiple approaches and what they are and WHY you chose your approach.

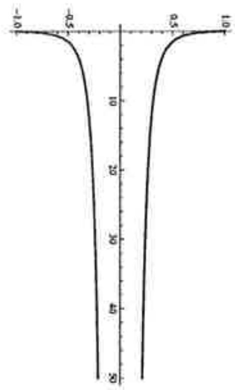


Figure 2: Analytic result for water flowing from a spigot.

When applicable: tell readers how to "check" your equations to show they make sense

At the top of the spigot all of the energy in the water is in the form of kinetic energy: $E = dmv_0^2/2$. After the water has fallen a distance h it has energy $E = dmv(h)^2/2 - dmgh$ (I know that the potential energy is $-mgh$ because the speed of the water must *increase* as we look further along the stream). Therefore, energy conservation dictates:

$$\frac{1}{2}dmv_0^2 = \frac{1}{2}dmv^2 - dmgh \rightarrow v(h) = \sqrt{v_0^2 + 2gh}. \tag{4}$$

clearly define ALL symbols

We can further write $A = \pi[w(h)/2]^2$, where $w(h)$ is the diameter of the stream's cross-sectional area. Combining Eq. (3) with Eq. (4) we have

$$\left(\frac{w(h)}{w_0}\right)^2 = \frac{v_0}{\sqrt{v_0^2 + 2gh}} \rightarrow \frac{w(h)}{w_0} = \left(\frac{1}{1 + 2gh/v_0^2}\right)^{1/4}. \tag{5}$$

Taking $g = 10$ m/s^2 and $v_0 = 10$ cm/s (which seems like a reasonable speed) we can now plot how

plug in reasonable #'s if applicable

the width of the stream decreases with distance from the spigot. I have shown that in Fig. 2 We can see that the width of the stream is qualitatively similar to the actual picture in Fig. 1.