1. Why the cross product has no inverse.

For all vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$, we can write the cross product $\boldsymbol{a} \times \boldsymbol{b}$ as a matrix operation

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b}$$

where $[\boldsymbol{a}]_\times$ is a *skew-symmetric* matrix depending only on $\boldsymbol{a}$. A matrix $\boldsymbol{A}$ is skew-symmetric if

$$\boldsymbol{A}^T = -\boldsymbol{A}$$

a. What are the elements of $[\boldsymbol{a}]_\times$? Note that each element of $(\boldsymbol{a} \times \boldsymbol{b})$ is linear in the elements of $\boldsymbol{b}$. What are the coefficients?

b. Show that $[\boldsymbol{a}]_\times$ has a non-trivial null space. You can do this either by computing $\det([\boldsymbol{a}]_\times)$ or by showing that one of its columns can be expressed as a linear combination of the other two.

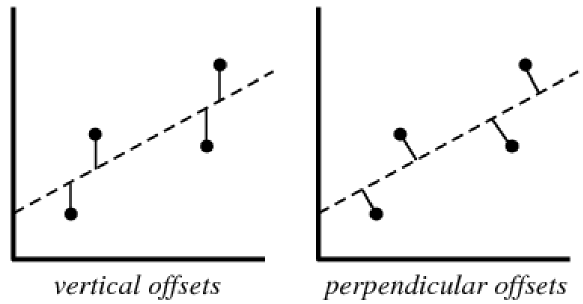
c. Explain why the cross product has no inverse. That is, for a given vector $\boldsymbol{c}$, why is there no unique vector $\boldsymbol{b}$ such that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$?

2. Lines and homogenous coordinates.

a. Line between two points. In class, we saw that the intersection of two lines $\tilde{\boldsymbol{\ell}}_1$ and $\tilde{\boldsymbol{\ell}}_2$ can be computed by $\tilde{\boldsymbol{\ell}}_{\boldsymbol{1}} \times \tilde{\boldsymbol{\ell}}_{\boldsymbol{2}}$. Similarly, the line connecting two points $\tilde{\boldsymbol{p}}_1$ and $\tilde{\boldsymbol{p}}_2$ can be computed by $\tilde{\boldsymbol{p}}_1 \times \tilde{\boldsymbol{p}}_2$. Prove this using properties of the dot product and the cross product. *Hint: the proof should be trivial.*

b. Distance from a point to a line. Given the line $\tilde{\boldsymbol{\ell}} = (a, b, c)$ with $a^2 + b^2 = 1$, show that for any point $\tilde{\boldsymbol{p}}$, the *signed distance* from the point to the line is computed by $d = \tilde{\boldsymbol{\ell}} \cdot \bar{\boldsymbol{p}}$, where the sign of d indicates which side of the line the point is on, and the magnitude is equal to the perpendicular distance from $\tilde{\boldsymbol{p}}$ to $\tilde{\boldsymbol{\ell}}$.

c. Least squares line fitting with perpendicular distances. Least squares line fitting typically minimizes the vertical offsets between a line and a set of points. Using the homogeneous least squares technique we discussed in class, derive a method to minimize the *perpendicular* offsets instead.



http://mathworld.wolfram.com/LeastSquaresFittingPerpendicularOffsets.html

Given n augmented points $\bar{\boldsymbol{p}}_1, \ldots, \bar{\boldsymbol{p}}_n$, your method should seek to find the line $\tilde{\boldsymbol{\ell}}$ that minimizes the residual

$$\sum_{i=1}^{n} \left(\tilde{\boldsymbol{\ell}} \cdot \bar{\boldsymbol{p}}_i \right)^2$$

subject to $\|\tilde{\boldsymbol{\ell}}\| = 1$.

3. Rigid transformations.

A 2D rigid transformation is an invertible transformation which preserves distances. For any point $\boldsymbol{p} \in \mathbb{R}^2$, we can write the transformation as

$$\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p} + \boldsymbol{t}$$

where $\boldsymbol{R}$ is a 2×2 rotation matrix and $\boldsymbol{t}$ is a translation vector.

a. Matrix representation. Show that the transformation can be represented as a 3×3 homogenous matrix $\tilde{\boldsymbol{M}}$ such that $\bar{\boldsymbol{p}}' = \tilde{\boldsymbol{M}}\bar{\boldsymbol{p}}$. What is the matrix?

b. Matrix inverse. Solve for $\boldsymbol{p}$ in terms of $\boldsymbol{R}$, $\boldsymbol{t}$, and $\boldsymbol{p}'$. What is $\tilde{\boldsymbol{M}}^{-1}$?

c. Rigid transformation of a line. Given a rigid transformation specified by $(\boldsymbol{R}, \boldsymbol{t})$ and a line $\tilde{\boldsymbol{\ell}}$, what is the corresponding line $\tilde{\boldsymbol{\ell}}'$ such that for all $\bar{\boldsymbol{p}}$,

$$\tilde{\boldsymbol{\ell}}' \cdot \bar{\boldsymbol{p}}' = \tilde{\boldsymbol{\ell}} \cdot \bar{\boldsymbol{p}}$$

Express $\tilde{\boldsymbol{\ell}}'$ in terms of $\tilde{\boldsymbol{M}}$ and $\tilde{\boldsymbol{\ell}}$.