Math 25 Worksheet 13 - Power Series 1

1. Write each series below using sigma notation.

(a) $1 + \frac{3}{2!} + \frac{9}{4!} + \frac{27}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{3^n}{(2n)!}$

(b) $\frac{(x-a)}{3} + \frac{(x-a)^2}{9 \cdot 2!} + \frac{(x-a)^3}{27 \cdot 3!} + \cdots = \sum_{n=1}^{\infty} \frac{(x-a)^n}{3^n\, n!}$

(c) $\frac{(x-3)}{2!} - \frac{(x-3)^3}{4!} + \frac{(x-3)^5}{6!} - \frac{(x-3)^7}{8!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{(2n)!}$

2. Write the first two or three terms of each series below.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n!)} = \frac{1}{2 \cdot 1!} - \frac{1}{2 \cdot 2!} + \frac{1}{2 \cdot 3!} - \cdots$ (note the difference between $2n!$ and $(2n)!$)

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$

(c) $\sum_{n=0}^{\infty} \frac{2^n x^n}{n^3+1} = 1 + \frac{2x}{1+1} + \frac{2^2 x^2}{2^3+1} + \cdots$ (It's usually ok to leave expressions unsimplified.)

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1} = \frac{1}{4} - \frac{1}{7} + \frac{1}{10} - \cdots$ (It's also ok to do the arithmetic when it's easy.)

(e) $\sum_{n=0}^{\infty} \frac{(n!)^2 (x-1)^n}{(2n)!} = 1 + \frac{(x-1)}{2!} + \frac{(2!)^2 (x-1)^2}{4!} + \cdots$

3. Use the Ratio Test to determine the intervals in which series 1c, 2b, and 2e converge. Ignore the endpoints for now

1c $\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\frac{(x-3)^{n+1}}{(2(n+1))!}}{\frac{(x-3)^n}{(2n)!}} \right| = \left| \frac{(2n)!}{(2n+2)!}(x-3) \right| = \frac{|x-3|}{(2n+2)(2n+1)}$. As $n \to \infty$, this expression $\to 0$ for all x.

So this series converges for all x (has radius of conv. $= \infty$.)

2b $\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \left| \frac{n}{n+1} x \right|$.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1} x\right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|$.

So by Ratio Test, the series converges for $|x| < 1$ and has radius of conv. $= 1$.

Checking endpts;

$x = 1$: $\sum \frac{(-1)^{n-1}}{n}$ converges by AST.

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$

$= -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$; diverges as negative of Harmonic Series.

So the interval of convergence for this series is $(-1, 1]$ or $-1 < x \le 1$.

(2e) $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{((n+1)!)^2 (x-1)^{n+1}}{(2(n+1))!}}{\dfrac{(n!)^2 (x-1)^n}{(2n)!}}\right|$$

$$= \left|\frac{(n+1)!\,(n+1)!\,(2n)!}{n!\,n!\,(2n+2)!}(x-1)\right| = \left|\frac{(n+1)^2}{(2n+2)(2n+1)}(x-1)\right|$$

So $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}|x-1| = |x-1|\lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2}$$

$= |x-1|\left(\frac{1}{4}\right)$ which is < 1 for $|x-1| < 4$.

So radius of conv. is 4 :

And the series converges for $-3 < x < 5$, at least.
We skip the endpoints as this series is overly messy there, and going through details is not illuminating.