Math 25
Spring 2016

Practice Exam 1 Solutions

1) Let $u = \sin(2x)$. Then $du = 2\cos(2x)\,dx$. So $\frac{1}{2}du = \cos(2x)dx$.

Thus $\int_0^{\pi/2} e^{\sin(2x)}\cos(2x)dx = \frac{1}{2}\int_{x=0}^{x=\pi/2} e^u du = \frac{1}{2} e^u\Big]_{x=0}^{x=\pi/2}$

$= \frac{1}{2} e^{\sin(2x)}\Big]_0^{\pi/2} = \frac{1}{2}\left[e^{\sin\pi} - e^{\sin 0}\right] = \frac{1}{2}\left[e^0 - e^0\right] = 0.$

2) Consider $\frac{2x+1}{2x^2+4x} = \frac{2x+1}{2x(x+2)} = \frac{A}{2x} + \frac{B}{x+2} = \frac{A(x+2)+B(2x)}{2x(x+2)}$.

Equating our numerators yields $2x+1 = A(x+2)+B(2x)$.

when $x=0$: $1 = 2A$, so $A = 1/2$

when $x=-2$: $-3 = -4B$, so $B = 3/4$.

Thus, $\int \frac{2x+1}{2x^2+4x}dx = \int \frac{1/2}{2x}dx + \int \frac{3/4}{x+2}dx = \frac{1}{4}\ln|x| + \frac{3}{4}\ln|x+2| + C.$

3) let $u = e^t$. Then $du = e^t dt$. And $1+e^{2t} = 1+(e^t)^2$.

So $\int \frac{e^t}{1+e^{2t}}dt = \int \frac{1}{1+(e^t)^2}e^t dt = \int \frac{1}{1+u^2}du = \arctan u + C$

$= \arctan(e^t) + C.$

4) we find $\int \sqrt{x}\ln x\,dx$ using integration by parts: Let $u = \ln x$, $dv = x^{1/2}$

$du = \frac{1}{x}dx$, $v = \frac{2}{3}x^{3/2}$

So $\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \int \underbrace{\frac{1}{x}\cdot\frac{2}{3}x^{3/2}}_{=\frac{2}{3}x^{1/2}}dx$

$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C.$

Thus $\int_1^4 \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\left(\ln x - \frac{2}{3}\right)\Big]_1^4 = \left.\begin{matrix}\frac{2}{3}4^{3/2}\left(\ln 4 - 2/3\right)\\ -\frac{2}{3}\left(\ln 1 - 2/3\right)\end{matrix}\right\} (*)$

$= \frac{2}{3}(8)(\ln 4 - 2/3) - \frac{2}{3}\left(-\frac{2}{3}\right) = \frac{16}{3}\ln 4 - \frac{32}{9} + \frac{4}{9} = \frac{16}{3}\ln 4 - \frac{28}{9}.$

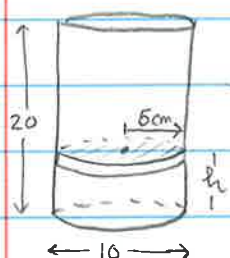
(note that you could leave this answer as it is at *.)

5) To find $\int \frac{x}{(1+x^2)^3}\,dx$, let $u=(1+x^2)$. So $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$.

Thus our integral becomes $\frac{1}{2}\int u^{-3}\,du = -\frac{1}{4}u^{-2}+C = \frac{-1}{4(1+x^2)^2}+C.$

Now, $\int_1^{\infty}\frac{x}{(1+x^2)^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{(1+x^2)^3}\,dx = \lim_{b\to\infty}\left[\frac{-1}{4(1+x^2)^2}\right]_1^b$

$= \lim_{b\to\infty}\left[\frac{-1}{4(1+b^2)^2} - \frac{-1}{4(1+1)^2}\right] = 0 + \frac{1}{16} = \frac{1}{16}$

6) 

Slice Δh cm thick, h cm from the bottom.

Vol. of this slice $= \pi(5)^2\Delta h$ cm^3 $= 25\pi\Delta h$

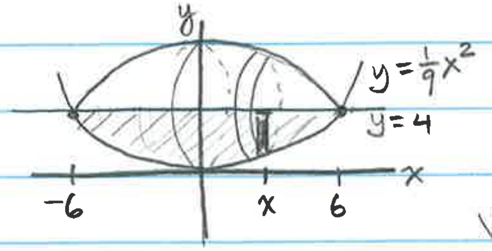
Density of sugar in the slice is $\approx 3(20-h)$ mg/cm^3,

so the mass of sugar in slice is $\approx 3(20-h)25\pi\Delta h$ mg.

Thus, total mass $= \int_0^{20} 3(20-h)(25\pi)\,dh = 75\pi\int_0^{20} 20-h\,dh$

$= 75\pi\left(20h-\frac{1}{2}h^2\right)\Big]_0^{20} = 75\pi\left[(400-200)-(0-0)\right] = 15{,}000\pi$ mg.

yikes!

7) 

The region being revolved is shaded.////

Slice perpendicular to x-axis, with thickness Δx.

Vol. of slice at x is $\approx \pi r^2\Delta x$,

where r is the distance from line $y=4$ to curve $y=\frac{1}{9}x^2$.

Thus, $r = 4-\frac{x^2}{9}$, So vol. of slice is $\approx \pi\left(4-\frac{x^2}{9}\right)^2\Delta x$.

(a) Vol. of ball $\approx \sum \pi\left(4-\frac{x^2}{9}\right)^2\Delta x$.

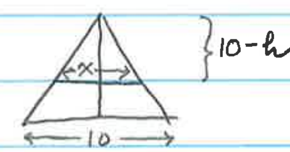
(b) Actual volume $= \int_{-6}^{6}\pi\left(4-\frac{x^2}{9}\right)^2 dx = 2\int_0^6 \pi\left(4-\frac{x^2}{9}\right)^2 dx$, by symmetry.

$= 2\pi\int_0^6\left(16-\frac{8}{9}x^2+\frac{1}{81}x^4\right)dx = 2\pi\left(16x-\frac{8}{27}x^3+\frac{1}{5\cdot 81}x^5\right)\Big]_0^6$

$= 2\pi\left[\left(16\cdot 6-\frac{8}{27}(6)^3+\frac{1}{5\cdot 81}6^5\right)-0\right].$

8) Slice pyramid at height h, parallel to base, with thickness Δh. Vol of slice $\approx$ (area of top of slice) $\cdot \Delta h$

Top of slice is a square, say x by x,
where $\frac{x}{10} = \frac{10-h}{10}$, by similar Δ's:



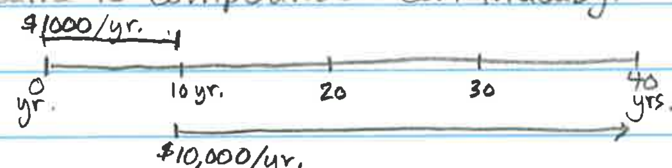
So $x = 10-h$.

Thus, volume of a slice is $\approx (10-h)^2 \Delta h$.

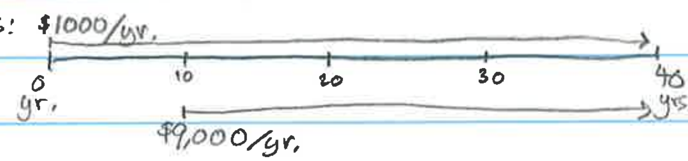
Therefore, vol. of pyramid $= \int_0^{10} (10-h)^2 \, dh$.

9) Interest rate is 5%, which we assume is compounded continuously.

Here's a schematic of income stream:



The deposits made in first 10 years continue to grow, so we can reconfigure as follows:



Thus, the retirement nest egg after 40 years will equal the future value of \$1000 stream for 0 to 40th yrs. plus future value of \$9000 stream for 10th to 40th years, both at 5%.

So
$$B = \int_0^{40} 1000\, e^{(.05)(40-t)}\, dt + \int_{10}^{40} 9000\, e^{(.05)(40-t)}\, dt$$

(Note that second integral can also be written as $\int_0^{30} 9000\, e^{(.05)(30-t)}\, dt$.)

To evaluate, note that $e^{(.05)(40-t)} = e^{2-(.05)t} = e^2 e^{-(.05)t}$.

So $\int e^{(.05)(40-t)}\, dt = \int e^2 e^{-(.05)t}\, dt = e^2 \cdot \frac{e^{-(.05)t}}{-(.05)} + C = -20 e^2 e^{-(.05)t} + C$

So $\int_0^{40} 1000\, e^{(.05)(40-t)}\, dt = -20{,}000\, e^2 \left[e^{-(.05)t}\right]_0^{40} = -20{,}000\, e^2 \left[e^{-2} - 1\right]$

and $\int_{10}^{40} 9000\, e^{(.05)(40-t)}\, dt = -180{,}000\, e^2 \left[e^{-(.05)t}\right]_{10}^{40} = -180{,}000\left[e^{-2} - e^{-.5}\right]$

B = sum of these two quantities (which happens to equal $\approx$ \$754,485).